

Béla Bollobás

# THE Art OF Mathematics

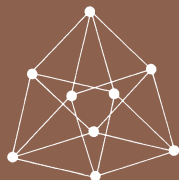
Coffee Time in Memphis

Bollobás  
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Can a Christian escape from a lion? How quickly can a rumour spread? Can you fool an airline into accepting oversize baggage? Recreational mathematics is full of frivolous questions in which the mathematician's art can be brought to bear. But play often has a purpose, whether it's bear cubs in mock fights, or war games. In mathematics, it can sharpen skills, or provide amusement, or simply surprise, and collections of problems have been the stock-in-trade of mathematicians for centuries. Two of the twentieth century's greatest players of problem posing and solving, Erdős and Littlewood, are the inspiration for this collection, which is designed to be sipped from, rather than consumed, in one sitting. The questions themselves range in difficulty: the most challenging offer a glimpse of deep results that engage mathematicians today; even the easiest are capable of prompting readers to think about mathematics. All come with solutions, many with hints, and most with illustrations. Whether you are an expert, or a beginner, or an amateur, this book will delight for a lifetime.

BÉLA BOLLOBÁS is a Senior Research Fellow at Trinity College, Cambridge and holds the Jabie Hardin Chair of Excellence in Combinatorics at the University of Memphis. He has held visiting positions from Seattle to Singapore, from Brazil to Zurich. This is his tenth book.

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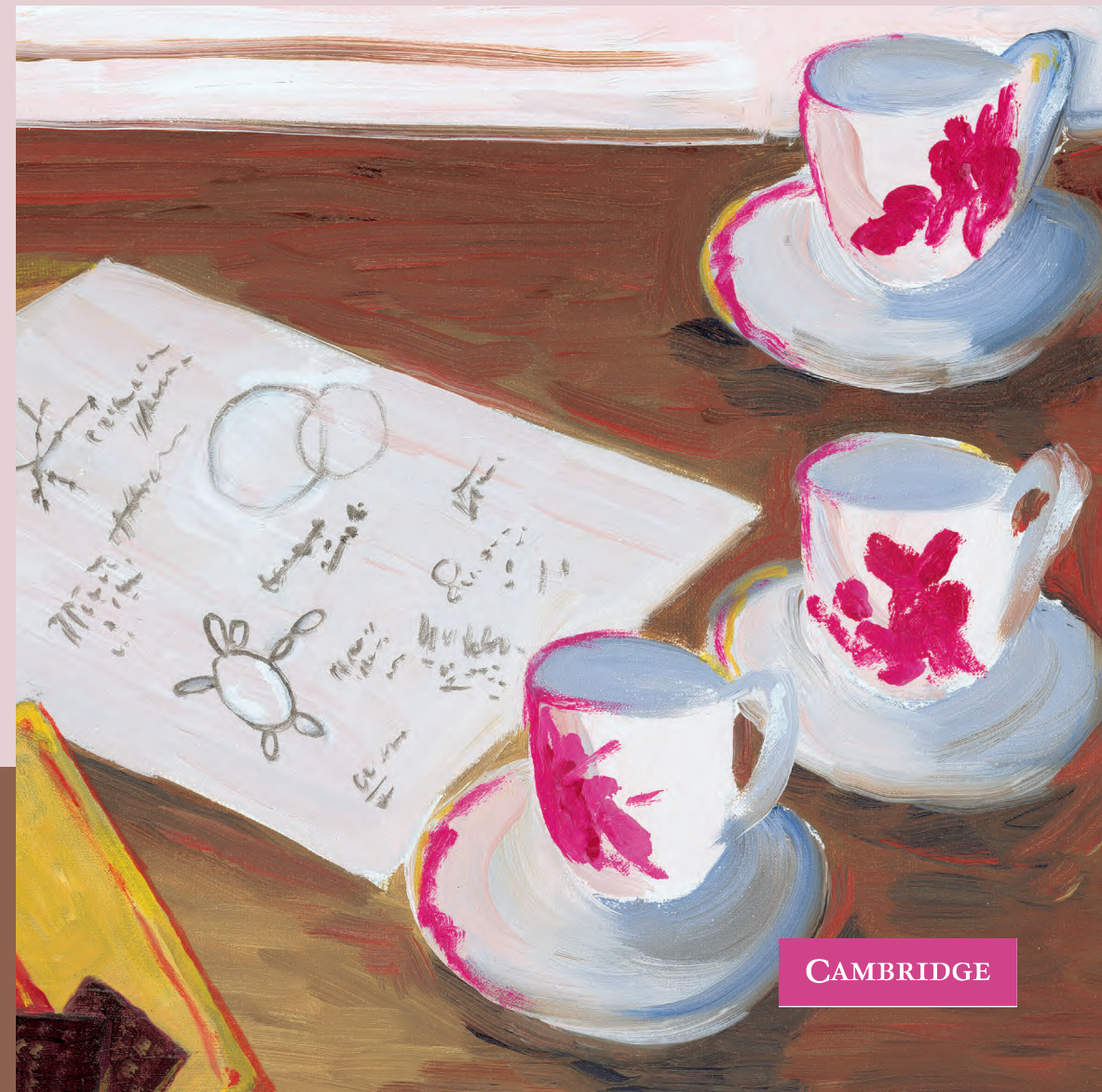


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GABRIELLA BOLLOBÁS is an eminent portraitist. She has sculpted busts of many outstanding scientists, including Dirac, von Neumann, Hardy, Littlewood, Erdős and Hawking. She lives in Cambridge with her husband.

*In re mathematica ars proponendi quaestionem pluris facienda est quam  
solvendi.*

Georg Cantor



*Coffee Time in Memphis*

# The Art of Mathematics

## *Coffee Time in Memphis*

Béla Bollobás  
*University of Memphis and  
University of Cambridge*



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*To*  
*J.E. Littlewood (1885–1977)*  
*and*  
*Paul Erdős (1913–1996)*



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## Preface

When I was putting together this collection of problems, I always asked myself whether the two giants of mathematics I had the good fortune to know well, Paul Erdős and J.E. Littlewood, would have found the question interesting. Would they have felt enticed to think about it? Could they have *not* thought about it, whether they wanted to or not? I think that many of the problems that ended up in this volume are indeed of the kind Erdős and Littlewood would have found difficult not to think about; since this collection contains many problems they considered or even posed, this assertion may not be as preposterous as it seems.

I was not yet ten when I fell in love with mathematical problems. Growing up in Hungary, this love got plenty of encouragement, and when at fourteen I got to know Paul Erdős, the greatest problem poser the world has ever seen, my fate was sealed. He treated me and other young people to a variety of beautiful and fascinating problems, solved and unsolved; many of the solved ones I heard from him in my teens appear in this volume.

The impetus for putting together this collection of problems came much later, in Memphis, where, for a few years now, some of the local and visiting mathematicians have had the habit of having lunch together, followed by coffee and a mathematical problem or two in my office. After a while, this meeting became quite an institution, with the tacit understanding that I would provide espresso and chocolate to please the palate, and a problem or two for the delectation of the mind. A *sine qua non* was that the problem should be enjoyable. The problems arising at these sessions form the core of this collection, so that for many years the working title of this book was *Coffee Time in Memphis*. It was only when I came to publishing it that my friend David Tranah, and his colleagues at CUP, insisted on changing the somewhat frivolous title I suggested to its present lofty incarnation, with CTM relegated to the subtitle.

It should be emphasized that this is a rather haphazard collection: in addition to the original ‘coffee time’ problems, and those from Erdős and Littlewood, there are many frivolous mathematical puzzles and a few problems from my teaching days in Cambridge, where I used to produce example sheets *For the enthusiast*. All of them are *mathematics with fun*: this is the main reason for publishing them in a volume. But just as a bear-cub can acquire life skills through play, so the reader can learn skills for a mathematical life simply by solving or just trying to solve the problems. The uninitiated can get a glimpse of how mathematical research is done and how the

mathematician's art is a combination of taste and technique; the professional can test his agility and ingenuity, 'the temper of his steel', as Hilbert said. Several of the problems are milestones in themselves or introduce the reader to serious areas of mathematics.

A few words of warning. This book is not a collection of *The Hundred Most Beautiful Theorems*, nor is it suitable for a systematic study of mathematics. Reading the book from cover to cover, as if it were an introduction to a branch of mathematics, may well cause indigestion and more damage than good.

Ideally, this collection should be used as a source of 'coffee time', enjoyable problems. The reader should pick out a problem or two to think about: if the problem is solved easily, fine, the next problem can come; but if it resists the initial attacks, the reader is likely to be even better off, for then the eventual solution (whether read or discovered) will be more pleasurable and beneficial. In particular, it is hoped that many of the questions can be used to inspire undergraduates taking standard, main-line, courses in mathematics.

The volume consists of three unequal parts. In the first part, over hundred and fifty problems are given, and the very brief second part contains hints to some of these problems. The third part is the longest by far: here the problems tend to be stated as theorems, and the solutions are given as proofs. Most solutions are followed by notes giving more information and references about the results. Although I have tried to give quite a few references, I have no doubt that some attributions should be more accurate and others are entirely missing: I apologize for these shortcomings. I shall be happy to correct any inaccuracies that brought to my attention.

Finally, it is a pleasure to acknowledge the help I have received in producing this book. First of all I should like to thank the 'regulars' and frequent visitors at coffee: Paul Balister, the mainstay of the round table, Stephen Kalikow, Vlado Nikiforov, Anthony Quas, Oliver Riordan, Amites Sarkar and Mark Walters, and the occasional visitors: Graham Brightwell, András Gyárfás, Ervin Győri, Imre Leader, Charles Read, Alex Scott, Miklós Simonovits, and many others. Without their love of mathematics and mathematical problems, this project would never even have been started. I should like to thank Paul Balister, József Balogh, Jonathan Cutler, Robert Morris, Vlado Nikiforov and Amites Sarkar for reading parts of the manuscript and helping me weed out a number of mistakes; for the many that no doubt remain, I apologize. For producing most of the figures, I am grateful to my invaluable and tireless assistant, Mrs. Tricia Simmons. Finally, I am most grateful to my wife, Gabriella, for putting the 'Art' into this book.

Béla Bollobás

*Cambridge, St. George's Day, 2006.*

# 1

## The Problems

**1. The Lion and the Christian.** A lion and a Christian in a closed circular Roman arena have equal maximum speeds. What tactics should the lion employ to be sure of his meal? In other words, can the lion catch the Christian in finite time?

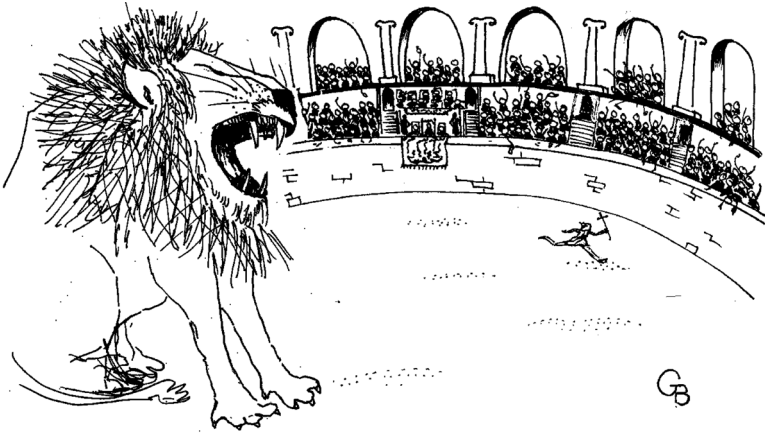


Fig. 1. A Roman lion.

## 2. Integer Sequences

- (i) Show that among  $n + 1$  positive integers none of which is greater than  $2n$  there are two such that one divides the other.
- (ii) Show that among  $n + 1$  positive integers none of which is greater than  $2n$  there are two that are relatively prime.



- (iii) Suppose that we have  $n$  natural numbers none of which is greater than  $2n$  such that the least common multiple of any two is greater than  $2n$ . Show that all  $n$  numbers are greater than  $2n/3$ .
- (iv) Show that every sequence of  $n = rs + 1$  distinct integers with  $r, s \geq 1$  has an increasing subsequence of length  $r + 1$  or a decreasing subsequence of length  $s + 1$ .

### 3. Points on a Circle

- (i) Let  $X$  and  $Y$  be subsets of the vertex set of a regular  $n$ -gon. Show that there is a rotation  $\varrho$  of this polygon such that  $|X \cap \varrho(Y)| \geq |X||Y|/n$ , where, as usual,  $|Z|$  denotes the number of elements in a finite set  $Z$ .
- (ii) Let  $S = F_1 \cup F_2$ , where  $F_1$  and  $F_2$  are closed subsets of  $S$ , the unit circle in  $\mathbb{R}^2$ . Show that either  $F_1$  or  $F_2$  is such that, for every angle  $0 < \alpha \leq \pi$ , it contains two points that form angle  $\alpha$  with the centre.
- (iii) A set  $S$  of integers is *sum-free* if  $x + y = z$  has no solution in  $S$ , i.e.  $x + y \notin S$  whenever  $x, y \in S$ . Show that every set of  $n \geq 1$  non-zero integers contains a sum-free subset of size greater than  $n/3$ .

**4. Partitions into Closed Sets.** Can the plane be partitioned into countably many non-empty closed sets? And what about  $\mathbb{R}^n$ ?

**5. Triangles and Squares.** What is the maximal area of a triangle contained in a unit square? And the minimal area of a triangle containing a unit square?

**6. Polygons and Rectangles.** Show that every convex polygon of area 1 is contained in a rectangle of area 2.

**7. African Rally.** A car circuit goes through the towns  $T_1, T_2, \dots, T_n$  in this cyclic order (so from  $T_i$  the route leads to  $T_{i+1}$  and from  $T_n$  back to  $T_1$ ), and a car is to travel around this circuit, starting from one of the towns. At the start of the journey, the tank of the car is empty, but in each town  $T_i$  it can pick up  $p_i$  amount of fuel.

- (i) Show that if  $\sum_{i=1}^n p_i$  is precisely sufficient to drive round the entire circuit then there is a town such that if the car starts from there then it can complete the entire circuit without running out of fuel.
- (ii) Show that if each section of the circuit (from town  $T_i$  to  $T_{i+1}$ ) needs an integer amount of fuel, each  $p_i$  is an integer and  $\sum_{i=1}^n p_i$  is precisely 1 more than the amount of fuel needed to drive round the entire course, then there is precisely one town such that, starting from there, there is at least 1 unit of fuel in the tank not only at the end but throughout the circuit.

**8. Fixing Convex Domains.** A convex board is surrounded by some nails hammered into a table: the nails make impossible to slide the board in any direction, but if any of them is missing then this is no longer true. What is the maximal number of nails?

**9. Nested Subsets.** Is an infinite family of *nested* subsets of a countable set necessarily countable?

**10. Almost Disjoint Subsets.** Call two sets *almost disjoint* if their intersection is finite. Is there an uncountable family of almost disjoint subsets of a countable set?

**11. Loaded Dice.** We have two loaded dice, with  $1, 2, \dots, 6$  coming up with various (possibly different) probabilities on each. Is it possible that when we roll them both, each of the sums  $2, 3, \dots, 12$  comes up with the same probability?

**12. An Unexpected Inequality.** Let  $a_1, a_2, \dots$  be positive reals. Show that

$$\limsup_{n \rightarrow \infty} \left( \frac{1 + a_{n+1}}{a_n} \right)^n \geq e.$$

Show also that the inequality need not hold with  $e$  replaced by a larger number.

**13. Colouring Lines.** Let  $\mathcal{L}$  be a collection of  $k$ -element subsets of a set  $X$ . We call the elements of  $X$  *points*, and the  $k$ -subsets belonging to  $\mathcal{L}$  *lines*. ‘Maker’ and ‘Breaker’ play a game by alternately colouring the points of  $X$  *red* and *blue*, with Breaker making the first move. Once a point is coloured, it is never recoloured. Maker aims to ‘make’ a *red line*, a line all whose points are coloured red, and Breaker wants to prevent this. Show that if there are  $2^k - 1$  lines then Breaker has a winning strategy, but for some arrangement of  $2^k$  lines Maker has a winning strategy. What are the corresponding numbers if Maker makes the first move?

**14. Independent Sets.** Let  $G$  be a graph with vertex set  $V$ , and write  $d(v)$  for the degree of a vertex  $v$ . Show that  $G$  contains at least

$$\sum_{v \in V} \frac{1}{d(v) + 1}$$

independent vertices.

**15. Expansion into Sums  $2^i 3^j$ .** Every natural number may be expressed in binary form, i.e., is the sum of some numbers  $1, 2, 2^2, 2^3, \dots$ . In every such expansion, of any two of the summands one divides the other. Is it possible to write every natural number as a sum of numbers of the form  $2^i 3^j$  such that no summand divides the other? It is easily checked that the first few numbers have such representations; for example,  $19 = 4 + 6 + 9$ ,  $23 = 6 + 8 + 9$ ,  $115 = 16 + 27 + 72$ .

**16. A Tennis Match.** Alice and Bob are about to play a tennis match consisting of a single set. They decide that Alice will serve in the first game, and the first to reach twelve games wins the match (whether two games ahead or not). However, they are considering two serving schemes: the *alternating serves* scheme, in which the servers alternate game by game (how surprising!), and the *winner serves* scheme, in which the winner of a game serves in the next.

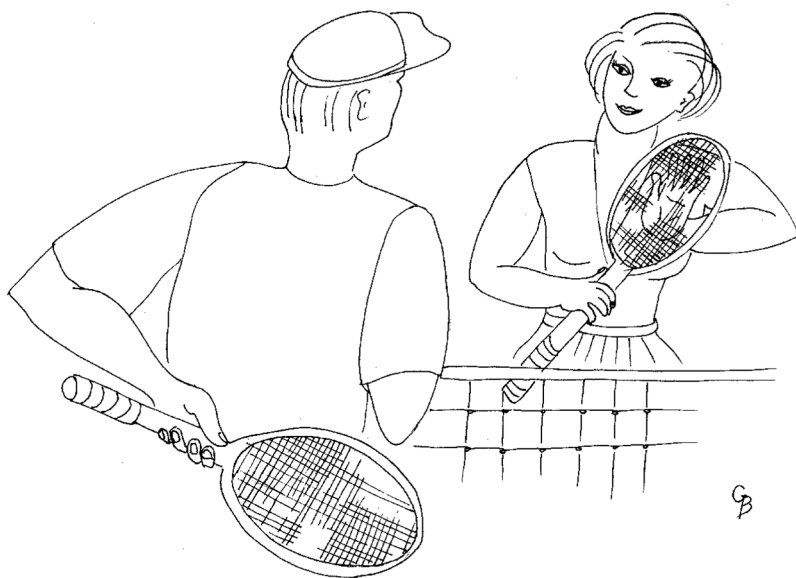


Fig. 2. The big question.

Alice estimates that she has 0.71 chance of winning her serve, while Bob has only 0.67 chance to hold serve. Which scheme should Alice choose to maximize her chances of winning?

**17. A Triangle Inequality.** Let  $R$  be the circumradius of an acute-angled triangle,  $r$  its inradius and  $h$  the length of the longest height. Show that  $r + R \leq h$ . When does equality hold?

**18. Planar Domains of Diameter 1.** Show that every planar domain  $D$  of diameter 1 is contained in a regular hexagon of width 1, i.e., side-length  $\sqrt{3}/3$ .

Deduce that every planar domain of diameter 1 can be partitioned into three sets of diameter at most  $\sqrt{3}/2$ .

**19. Orienting Graphs.** Show that for every graph there is an orientation of the edges such that for every vertex the out-degree and in-degree differ by at most 1.

**20. A Simple Clock.** How many times a day is it impossible to tell the time by a clock with identical hour and minute hands, provided we can always tell whether it is a.m. or p.m.?

**21. Neighbours in a Matrix.** Show that every  $n \times n$  matrix whose entries are  $1, 2, \dots, n^2$  in some order has two neighbouring entries (in a row or in a column) that differ by at least  $n$ .

**22. Separately Continuous Functions.** Let  $f : S = [0, 1]^2 \rightarrow \mathbb{R}$  be separately continuous in its variables, i.e., continuous in  $x$  for every fixed  $y$ , and continuous in  $y$  for every fixed  $x$ . Show that if  $f^{-1}(0)$  is dense in the square  $S$  then it is identically 0.

**23. Boundary Cubes.** A down-set in the solid  $d$ -dimensional cube  $[0, n]^d \subset \mathbb{R}^d$  is a set  $D \subset [0, n]^d$  such that if  $0 \leq x_i \leq y_i \leq n$  for  $i = 1, \dots, n$ , and  $\mathbf{y} = (y_i)_1^d \in D$  then  $\mathbf{x} = (x_i)_1^d \in D$  as well. What is the maximal number of boundary integral unit cubes, i.e., solid unit cubes

$$[c_1 - 1, c_1] \times [c_2 - 1, c_2] \times \cdots \times [c_d - 1, c_d],$$

with each  $c_i$  an integer,  $1 \leq c_i \leq n$ , that meets both a down-set  $D$  and its complement  $\overline{D}$ ?

**24. Lozenge Tilings.** A *lozenge* or *calisson* is a rhombus of side-length 1, with angles  $\pi/3$  and  $2\pi/3$ , i.e., the union of two equilateral triangles of side-length 1 that share a side. Figure 3 shows a tiling of a regular hexagon with lozenges.

Show that no matter how we tile a regular hexagon with lozenges, we must use the same number of tiles of each orientation.

**25. A Continuum Independent Set.** To every real number  $x$ , assign a finite set  $\Phi(x) \subset \mathbb{R} \setminus \{x\}$ . Call a set  $S \subset \mathbb{R}$  *independent* if  $x \in \Phi(y)$  does not hold for  $x, y \in S$ , that is if  $S \cap \Phi(S) = \emptyset$ . Show that there is an independent set  $S \subset \mathbb{R}$  whose cardinality is that of the continuum.

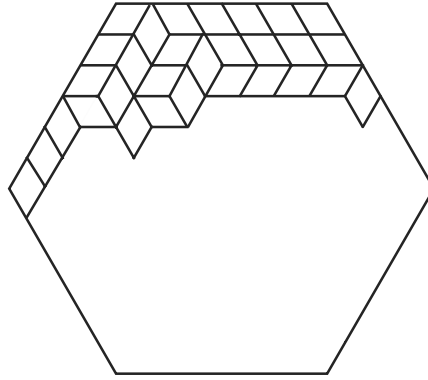


Fig. 3. The beginning of a tiling of the regular hexagon with lozenges.

**26. Separating Families of Sets.** Given an  $n$ -element set  $S$ , what is the minimal number of subsets  $S_1, \dots, S_k$  of  $S$  needed to separate all pairs of elements of  $S$ , i.e., so that for all  $x, y \in S$  there is a set  $S_i$  containing one of  $x$  and  $y$ , but not the other?

Equivalently, at least how many bipartite graphs are needed to cover (all the edges of) a complete graph on  $n$  vertices?

**27. Bipartite Covers of Complete Graphs.** Suppose that a family of bipartite graphs covers all the edges of a complete graph  $K_n$ . Show that altogether there are at least  $n \log_2 n$  vertices in these bipartite graphs.

More precisely, show that if  $n = 2^k + \ell < 2^{k+1}$ ,  $\ell \geq 0$ , then the minimum is precisely  $nk + 2\ell$ .

## 28. Convexity and Intersecting Simplices

- (i) Let  $X = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n+2}\}$  be a set of  $n + 2$  points in  $\mathbb{R}^n$  and, for a non-empty subset  $I$  of  $\{1, \dots, n + 2\}$ , let  $X(I)$  be the convex hull of the points  $\mathbf{x}_i$ ,  $i \in I$ . Show that there are disjoint sets  $I, J$  such that  $X(I) \cap X(J) \neq \emptyset$ .
- (ii) The convex hull  $\text{conv} X$  of a set  $X \subset \mathbb{R}^n$  is the smallest convex set containing  $X$ : the intersection of all convex sets containing it. Equivalently,

$$\text{conv} X = \left\{ \sum_{i=1}^k \lambda_i \mathbf{x}_i : \mathbf{x}_i \in X, \lambda_i \geq 0, \sum_{i=1}^k \lambda_i = 1, k = 1, 2, \dots \right\}.$$

Show that in the sums above we need not take more than  $n + 1$  terms, i.e., the convex hull of a set  $X \subset \mathbb{R}^n$  is

$$\text{conv}X = \left\{ \sum_{i=1}^{n+1} \lambda_i \mathbf{x}_i : \mathbf{x}_i \in X, \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1 \right\}.$$

**29. Intersecting Convex Sets.** Let  $\mathcal{C}$  be a finite family of convex sets in  $\mathbb{R}^n$  such that for  $k \leq n + 1$  any  $k$  members of  $\mathcal{C}$  have a non-empty intersection. Show that the intersection of *all* members of  $\mathcal{C}$  is non-empty.

**30. Judicious Partitions of Points.** Let  $P_1, \dots, P_n$  be  $n$  points in the plane. Show that there is a point  $P$  such that every line through  $P$  has at least  $n/3$  points  $P_i$  in each of the two closed half-planes it determines.

**31. Further Lozenge Tilings.** Consider a hexagon in which every angle is  $2\pi/3$  and the side-lengths are  $a_1, \dots, a_6$ . For what values of  $a_1, \dots, a_6$  is there a tiling of this hexagon with ‘lozenges’ of the three possible orientations, as in Problem 24; see Figure 4. How many lozenges are there of each orientation?

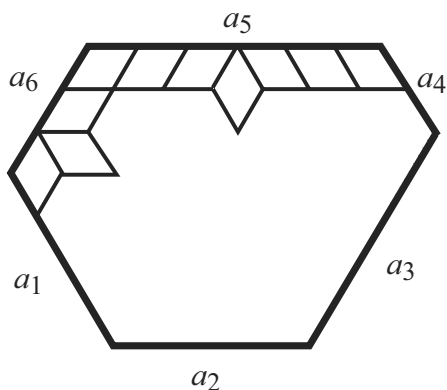


Fig. 4. A hexagon  $H(a)$ , with the beginning of a lozenge tiling.

**32. Two Squares in a Square.** Place two squares (with disjoint interiors) into a unit square. Show that the sum of the side-lengths is at most 1.

**33. Lines Through Points.** Let  $n$  points (of a Euclidean space) have the property that the line joining any two of them passes through a third point of the set. Must the  $n$  points all lie on one line?

**34. The Spread of Infection on a Square Grid.** A disease spreads on an  $n$  by  $n$  grid  $G_n$  as follows. At time 0 some of the *sites* (vertices, lattice points, grid points) are *infected*, the others are *healthy* but *susceptible*. Infected sites remain infected for ever, and at every time  $t$ ,  $t = 1, 2, \dots$ , every healthy site with at least two infected neighbours becomes itself infected. More formally, at time  $t = 0$  we have a set  $S_0$  of infected sites, and at each time step  $t$ ,  $t = 1, 2, \dots$ , we let  $S_t$  be the set of sites with at least two neighbours in  $S_{t-1}$ , together with the sites in  $S_{t-1}$ .  $S_t$  is the set of sites infected at time  $t$ . A set  $S_0$  of initially infected sites is said to *percolate* on the board  $G_n$ , with 2-neighbour infection or parameter 2, if, for some  $t$ ,  $S_t$  is the entire set of sites, i.e., if, eventually, every site becomes infected, as in Figure 5. What is the minimal number of initially infected sites that percolate?

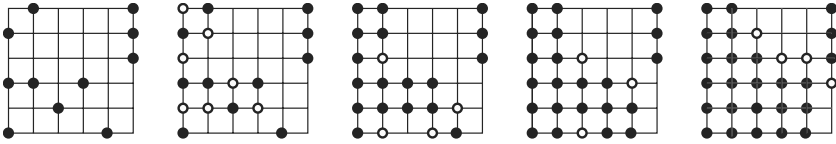


Fig. 5. The spread of a disease on a 6 by 6 grid, with the newly infected sites denoted by open circles  $\circ$ . At time  $t = 0$  we have 11 infected sites, and at time 4 there are 29. In another two steps all sites become infected.

**35. The Spread of Infection in a  $d$ -dimensional Box.** Let us consider the following extension of Problem 34. This time the disease spreads in a  $d$ -dimensional  $n \times n \times \dots \times n$  box

$$B = [n]^d = \{(x_i)_1^d \in \mathbb{Z}^d : 1 \leq x_i \leq n, i = 1, \dots, d\}$$

with  $n^d$  sites. As in Problem 34, the disease starts with a set  $S$  of infected sites in  $B$ . Every site with at least two infected neighbours becomes infected, and so is ready to infect other (neighbouring) sites. What is the minimal number of sites in  $S$  if eventually every site in  $B$  becomes infected?

**36. Sums of Integers.** Let  $1 \leq a_1 < a_2 < \dots < a_\ell \leq n$  be integers with  $\ell > (n + 1)/2$ . Show that  $a_i + a_j = a_k$  for some  $1 \leq i < j < k \leq \ell$ .

**37. Normal Numbers.** Given a real number

$$x = 0.x_1x_2\dots = \sum_{i=1}^{\infty} x_i 10^{-i}$$

written in base 10, and a sequence (word)

$$w = w_1w_2\dots w_k = \sum_{i=1}^k w_i 10^{k-i},$$

with terms (letters)  $0, 1, \dots, 9$ , write  $f_n(w; x)$  for the number of times  $x_1 x_2 \dots x_n$  contains  $w$ , i.e., the number of suffices  $i$ ,  $0 \leq i \leq n - k$ , such that  $k_j = w_{i+j}$  for  $j = 1, \dots, k$ . Note that if the letters of  $w$  are chosen at random (with every number  $0, 1, \dots, 9$  having probability  $1/10$ ) then the expectation of  $f_n(w; x)$  is  $(n - k + 1)10^{-k} \sim n10^{-k}$ .

A real number is *normal in base 10* if for every  $k$ -letter word  $w = w_1 w_2 \dots w_k$  with letters  $0, 1, \dots, 9$ , we have  $\lim_{n \rightarrow \infty} f_n(w; x)/n = 10^{-k}$ .

Let  $\gamma = 0.123 \dots 91011 \dots 99100101 \dots$  be the real number whose significant digits are formed by the concatenation of all the natural numbers. Show that  $\gamma$  is normal in base 10.

**38. Random Walks on Graphs.** Let  $s$  and  $t$  be vertices of a graph  $G$ . A random walk on  $G$  starts at  $s$  and stops at  $t$ . Show that the expected number of times this walk traverses the edges of a cycle in one direction is equal to the expected number of traversals in the other direction.

**39. Simple Tilings of Rectangles.** A tiling of a rectangle  $R$  with  $n \geq 2$  rectangles is called *simple* if no rectangle strictly inside  $R$  is a union of at least two rectangles of the tiling. For what values of  $n$  is there a simple tiling of a rectangle? Also, for large values of  $n$ , at least how many essentially different rectangles are there?

**40. L-tilings.** Cut out a square of a  $2^n$  by  $2^n$  chess board. Show that the remaining  $2^{2n} - 1$  squares can be tiled with  $L$ -tiles, where an  $L$ -tile is a union of three squares sharing a vertex, as the tiles in Figure 6.

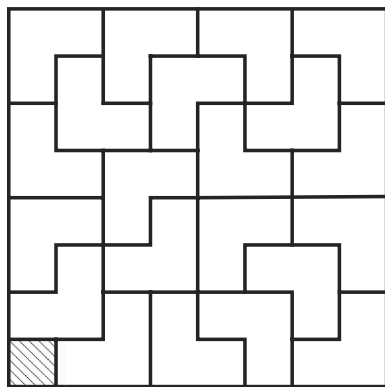


Fig. 6. An  $L$ -tiling of the  $8 \times 8$  board with the bottom left square cut out.



**41. Antipodal Points and Maps.** For  $D \subset \mathbb{R}^n$  and  $R \subset \mathbb{R}^m$ , a map  $f : D \rightarrow R$  is said to be *odd* if  $f(-x) = -f(x)$  for every  $x \in D$ . Two points  $x$  and  $y$  of the unit sphere  $S^n \subset \mathbb{R}^{n+1}$  are *antipodal* if  $y = -x$ . In view of this, an odd map  $S^n \rightarrow S^m$  is also said to be *antipodal*, since it maps antipodal points into antipodal points. Borsuk proved that continuous antipodal maps  $S^n \rightarrow S^n$  have odd degrees: in particular, they are not null-homotopic. This implies that

- (1) there is no continuous antipodal map  $S^n \rightarrow S^{n-1}$ .

Show that (1) is equivalent to each of the following three statements.

- (2) Every continuous map  $S^n \rightarrow \mathbb{R}^n$  sends at least one pair of antipodal points into the same point.  
 (3) Every continuous odd map  $S^n \rightarrow \mathbb{R}^n$  sends at least one point (and so at least one pair of antipodal points) into the origin of  $\mathbb{R}^n$ .  
 (4) For every family of  $n + 1$  closed sets covering  $S^n$ , one of the sets contains a pair of antipodal points.

**42. Bodies of Diameter 1.** Let  $K \subset \mathbb{R}^d$  be a body of diameter 1. Show that  $K$  is contained in a (closed) ball of radius  $r = \sqrt{d/(2d+2)}$ , and deduce that it can be partitioned into  $2^{d-1} + 1$  sets, each of diameter strictly less than 1.

**43. Equilateral Triangles.** Show that if equilateral triangles are erected externally (or internally) on the sides of a triangle as in Figure 7 then the centres of the new triangles form an equilateral triangle whose centroid is the centroid of the original triangle.

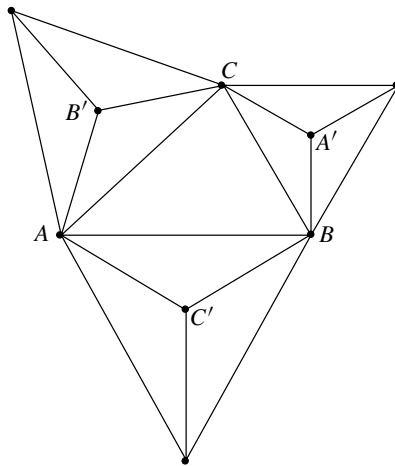


Fig. 7. Three equilateral triangles erected on a triangle  $ABC$ , with centroids  $A'$ ,  $B'$  and  $C'$ .

**44. Trisectors of Angles.** Prove that the trisectors of the angles of a triangle meet at the vertices of an equilateral triangle.

**45. Connected Subgraphs.** Show that in a graph of maximal degree  $d \geq 3$  there are at most  $(e(d-1))^n$  connected induced subgraphs with  $n+1$  vertices, one of which is a given vertex.

**46. Subtrees of an Infinite Tree.** Let  $D_2$  be the infinite rooted tree in which every vertex has two descendants. (Equivalently,  $D_2$  is the infinite tree in which one vertex, the root, has degree two, and every other vertex has degree three.) How many subtrees are there in  $D_2$  with  $n$  vertices, one of which is the root?

**47. Two-distance Sets.** Show that in the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  any  $\frac{1}{2}(n+1)(n+4)$  points determine at least three distances.

**48. Gossiping Dons.** Each of  $n$  Oxford dons has an item of gossip known only to himself. Whenever a don telephones another, they exchange all items of gossip they know. What is the minimal number of calls they have to make in order to ensure that every one of them knows all the gossip there is to know?



Fig. 8. A telephone call.

**49. Exact Covers.** Let  $\mathcal{L} = \{L_1, L_2, \dots, L_m\}$  be a collection of proper subsets of a set  $P = \{p_1, p_2, \dots, p_n\}$  of  $n$  elements (i.e.,  $|L_i| < n$  for every  $i$ ) such that any two elements of  $P$  are contained in exactly one set  $L_j$ . Show that  $m \geq n$ .

Show also that equality holds if and only if either one of the  $L_j$  contains  $n - 1$  elements  $p_i$ , and each of the other  $n - 1$  sets has two elements, or  $n$  is of the form  $n = k^2 + k + 1$ , each of the sets  $L_j$  consists of exactly  $k + 1$  elements, and any two of the sets  $L_j$  meet in exactly one element, i.e., if the sets  $L_j$  are the lines of a projective plane with point set  $\{p_1, \dots, p_n\}$ .

**50. Constant Intersections.** Let  $\mathcal{E} = \{E_1, E_2, \dots, E_n\}$  be a collection of subsets of  $[m] = \{1, \dots, m\}$  such that any two elements of  $\mathcal{E}$  intersect in exactly  $\lambda \geq 1$  elements. Show that  $n \leq m$ .

**51. Bell Numbers.** The  $n$ th Bell number  $B_n$  is the number of (unordered) partitions of  $[n] = \{1, 2, \dots, n\}$ . Thus  $B_1 = 1$ ,  $B_2 = 2$ ,  $B_3 = 5$  and  $B_4 = 15$ . For example, the set  $[3] = \{1, 2, 3\}$  has the following five partitions:  $\{123\}$ ,  $\{12, 3\}$ ,  $\{13, 2\}$ ,  $\{23, 1\}$  and  $\{1, 2, 3\}$ . Show that

$$B_n = \left\lceil \frac{1}{e} \sum_{k=1}^{2n} \frac{k^n}{n!} \right\rceil.$$

**52. Circles Touching a Square.** Figure 9 shows six non-overlapping circular discs of radius 1 touching a square of side-length 2. Can one construct seven such discs?

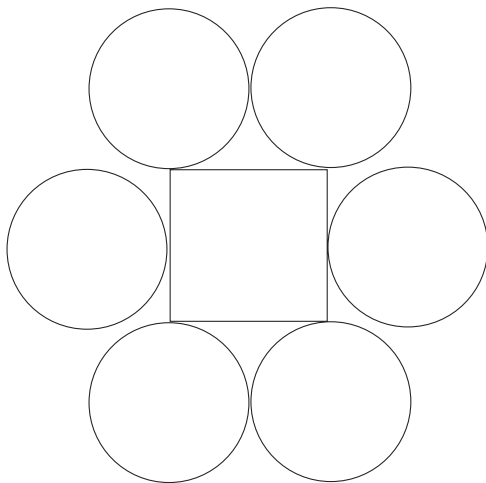


Fig. 9. Six circles touching a square.

**53. Gambling.** In a game of  $n$  gamblers, the  $i$ th gambler starts the game with  $a_i$  dollars. In each round, two gamblers selected at random make a fair bet, and the winner gets a dollar from the loser. A gambler losing all his money leaves the game. The game continues as long as possible, i.e., until one of the gamblers has all the money.

What is the expected number of rounds? What is the probability that the  $i$ th gambler ends up with all the money? For  $n = 3$ , what is the expected number of rounds played by all three players?

**54. Complex Sequences.** Let  $z_1, z_2, \dots$  be a sequence of complex numbers converging to 0. Is there a sequence of signs  $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1, \dots$ , for which the series  $\sum_{i=1}^{\infty} \varepsilon_i z_i$  is convergent?

### 55. Partitions of Integers.

- (i) Suppose that a number  $n$  can be partitioned into  $k$  distinct summands in which the maximum is  $m$ , and that there are  $p$  such summands. Show that  $p$  is also the number of partitions of  $n$  into  $m$  summands in which the summands are  $k, k-1, \dots, 1$ , each with multiplicity at least 1. For example, for  $n = 10, m = 5$  and  $k = 3$ , we have  $10 = 5 + 4 + 1, 10 = 5 + 3 + 2$ , and  $10 = 3 + 3 + 2 + 1 + 1, 10 = 3 + 2 + 2 + 2 + 1$  – two partitions each.

In particular, the number of partitions of an integer into distinct summands is the same as the number of its partitions into summands such that if  $s \geq 2$  appears as a summand then so does  $s - 1$ .

- (ii) Show that the number of partitions of an integer into distinct summands equals the number of its partitions into odd summands. For example, for  $n = 5$  we have the partitions  $5 = 5, 5 = 4 + 1, 5 = 3 + 2$ , and  $5 = 5, 5 = 3 + 1 + 1, 5 = 1 + \dots + 1$  – three partitions each.

**56. Emptying Glasses.** To keep little Jack occupied, his evil uncle gives him three large glasses, with each containing an integer number of ounces of water, and tells him to empty as many glasses as he can. However, all Jack is allowed to do is to double the contents of a glass from another, and keep repeating this operation. Thus, if at the beginning the three glasses contain  $a < b < c$  ounces, then each glass holds at least  $a + b + c$  ounces, and Jack is allowed to double  $a$  from  $b, a$  from  $c$ , and  $b$  from  $c$ , so from ‘state’  $(a, b, c)$  he can get to the states  $(2a, b - a, c)$ ,  $(2a, b, c - a)$  and  $(a, 2b, c - b)$ , and no other states.

What conditions should  $a, b$  and  $c$  satisfy to enable Jack to empty one of the glasses? And what are the conditions that make it possible to end up with all the water in one glass?



Fig. 10. A puzzled boy.

**57. Distances in Planar Sets.** Let  $P_1, \dots, P_n$  be  $n$  points in the plane.

- (i) Show that for some point  $P_i$ , the distances  $P_i P_j$ ,  $i \neq j$ , take at least  $(n - 3/4)^{1/2} - 1/2$  different values.
- (ii) At most how many times does the maximal distance occur among the points?
- (iii) And the minimal distance?

**58. Monic Polynomials.** For  $n \geq 1$ , let  $p_n(x)$  be a monic polynomial of degree  $n$  with real coefficients. Show that  $|p_n(x)| \geq 2$  for some  $x$  with  $-2 \leq x \leq 2$ . Show also that there need not be a point  $x$  with  $-2 \leq x \leq 2$  and  $|p_n(x)| > 2$ .

**59. Odd Clubs.** In *Oddtown*, with population  $n$ , every club has an odd number of members. Strangely, any two clubs have an even number of members in common. At most how many clubs are there in *Oddtown*?

**60. A Politically Correct Town.** In *Equaltown* there are  $n$  women and  $n$  men, and every club has the same number of women as men. All clubs have different sets of members and, amazingly, for any two clubs, exactly as many women as men belong to each. At most how many clubs are in *Equaltown*?

**61. Lattice Paths.** Call a lattice path on  $\mathbb{Z}^2$  with steps  $(0, 1)$  and  $(1, 0)$  an *increasing path*. How many increasing lattice paths are there from  $(r, s)$  to  $(m, n)$  that

never rise above the line  $y = x$ ? Putting it slightly differently, how many increasing walks are there on the lattice  $\mathbb{Z}^2$  from  $(r, s)$  to  $(m, n)$  that do not go above the line  $x = y$ ? Here  $s \leq r \leq m$  and  $s \leq n \leq m$ ; each step of an increasing walk goes from a lattice point  $(u, v)$  to either  $(u + 1, v)$  or  $(u, v + 1)$ , so that there are  $\binom{n+m-r-s}{m-r} = \binom{n+m-r-s}{n-s}$  increasing walks from  $(r, s)$  to  $(n, m)$ .

**62. Triangulations of Polygons.** In how many ways can a convex  $(n + 2)$ -gon be triangulated into  $n$  triangles by non-intersecting diagonals? Equivalently, in how many ways can one choose  $n - 1$  non-intersecting diagonals?

**63. A Converse of Cauchy's Inequality.** Recall that the  $L_1$ -norm of an integrable function  $\varphi : [0, \infty) \rightarrow \mathbb{R}$  is  $\|\varphi\|_1 = \int_0^\infty |\varphi(x)| dx$ . If  $\varphi, \psi : [0, \infty) \rightarrow \mathbb{R}$  are such that  $\varphi\psi$  is integrable, then

$$\langle \varphi, \psi \rangle = \int_0^\infty \varphi(x)\psi(x)dx.$$

Thus, if  $\varphi$  and  $\psi$  are non-negative then  $\langle \varphi, \psi \rangle = \|\varphi\psi\|_1$ .

Let  $\mathcal{D}$  be the set of decreasing functions  $f : [0, \infty) \rightarrow [0, \infty)$ , and  $\mathcal{U}$  the set of integrable functions  $F : [0, \infty) \rightarrow [0, 1]$ . Show that if  $f, g \in \mathcal{D}$  and  $F, G \in \mathcal{U}$  then

$$\langle f, g \rangle \max\{\|F\|_1, \|G\|_1\} \geq \langle f, F \rangle \langle g, G \rangle.$$

**64. Squares Touching a Square.** Figure 11 shows eight unit squares touching another. Can we have nine such squares?

**65. Infection with Three Neighbours.** As in Problem 34, there is an epidemic on the  $n \times n$  grid, but this time the sites are more resistant: a site becomes infected if at least three (rather than two) of its neighbours are infected. For  $n = 6k + 2$ , at least how many sites were infected at the start of an epidemic if eventually all  $n^2$  sites are infected?

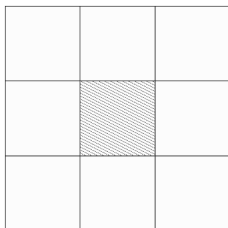


Fig. 11. A square touching eight others.

**66. The Spread of Infection on a Torus.** As in Problem 34, a disease spreads on a grid with the same infection rule, but this time the ‘board’ is an  $n$  by  $n$  grid on the torus. Putting it rather formally, we have  $n^2$  sites  $(i, j)$ ,  $0 \leq i, j \leq n-1$ , and a site  $(i, j)$  is adjacent to the four sites  $(i-1, j)$ ,  $(i+1, j)$ ,  $(i, j-1)$  and  $(i, j+1)$ , with the operations taken modulo  $n$ . (Thus, the neighbours of  $(1, n)$  are the sites  $(n, n)$ ,  $(2, n)$ ,  $(1, n-1)$  and  $(1, 1)$ .) Suppose that, eventually, every site becomes infected. At least how many sites were infected at the beginning?

**67. Dominating Sequences.** Let  $M_n$  be the set of strictly positive sequences of length  $n$ , and let  $M_n^*$  be the subset of monotone increasing sequences. Thus  $\mathbf{x} = (x_i)_1^n \in M_n^*$  if  $0 < x_1 \leq x_2 \leq \dots \leq x_n$ . We say that  $\mathbf{x} \in M_n$  *dominates*  $\mathbf{y} \in M_n$  (and write  $\mathbf{x} \geq \mathbf{y}$ ) if  $\sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i$  for every  $k$ . Finally, let  $\varphi(x)$  be a function on  $\mathbb{R}^+ = (0, \infty)$  and, for  $\mathbf{x} \in M_n$ , set  $S_\varphi(\mathbf{x}) = \sum_{i=1}^n \varphi(x_i)$ .

Show that if  $\varphi$  is a strictly convex decreasing function,  $\mathbf{x} \in M_n^*$ ,  $\mathbf{y} \in M_n$  and  $\mathbf{x} \geq \mathbf{y}$  then  $S_\varphi(\mathbf{x}) \leq S_\varphi(\mathbf{y})$ , with equality if and only if  $\mathbf{x} = \mathbf{y}$ .

**68. Sums of Reciprocals.** Let  $(a_i)_1^n$  be a sequence of natural numbers such that all  $2^n - 1$  sums formed from the  $a_i$  are distinct. (Equivalently,  $\sum_{i \in I} a_i \neq \sum_{j \in J} a_j$  whenever  $I \neq J \subset [n]$ .) Show that  $\sum_{i=1}^n 1/a_i < 2$ .

**69. Absent-minded Passengers.** A fully-booked plane is about to be boarded by its hundred passengers. The first passenger loses his boarding card and sits down in a random place. From then on, a passenger, whenever finding his seat taken, very politely sits somewhere at random. What is the probability that the hundredth passenger sits in his own seat?

**70. Airline Luggage.** Airlines frequently limit the size of the luggage one can check in by putting an upper bound on the *linear size* of the suitcase: *width plus length plus height*. Can you fool the airlines by packing your suitcase into a box of *strictly smaller linear size*?

Putting it into mathematical terms, write  $\ell(B) = a+b+c$  for the *linear measure* of an  $a \times b \times c$  box (rectangular parallelepiped). Is it true that if a box  $B$  is contained in a box  $C$  then  $\ell(B) \leq \ell(C)$ ? (Needless to say,  $B$  and  $C$  need not have the same ‘orientation.’)

**71. Intersecting Sets.** A family  $\mathcal{A}$  of sets is *intersecting* if  $A, B \in \mathcal{A}$  implies that  $A \cap B \neq \emptyset$ .

Let  $k$  and  $n$  be natural numbers with  $n \geq 2k$ , and let  $\mathcal{A} \subset X^{(k)}$  be an intersecting family of  $k$ -subsets of an  $n$ -element set  $X$ . For example, the set of all  $\binom{n-1}{k-1}$   $k$ -subsets of  $X$  containing a fixed element obviously has this property. Show that this

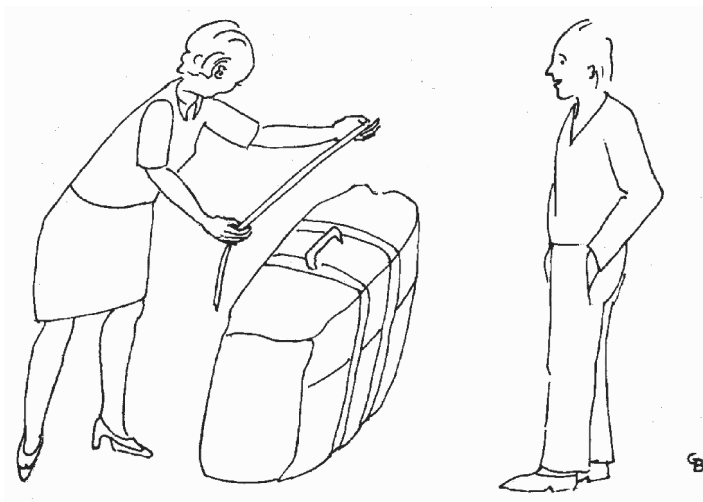


Fig. 12. How large is your suitcase?

system has the maximal number of elements, i.e.,

$$|\mathcal{A}| \leq \binom{n-1}{k-1}.$$

**72. Sperner Families.** A family of sets  $\mathcal{A}$  is a *Sperner family* if no set in  $\mathcal{A}$  is contained in another:  $A \not\subset B$  whenever  $A, B \in \mathcal{A}$ . Let  $\mathcal{A}$  be a Sperner family of subsets of a  $[n]$  (or any set with  $n$  elements). Show that

$$\sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1} \leq 1,$$

with equality if and only if for some  $k$ ,  $0 \leq k \leq n$ ,  $\mathcal{A} = [n]^{(k)}$ , i.e.,  $\mathcal{A}$  consists of all  $k$ -subsets of  $[n]$ .

In particular, a Sperner family of subsets of an  $n$ -element set consists of at most  $\binom{n}{k}$  sets.

**73. Breaking a Stick.** A stick is cut into  $n + 1$  pieces by  $n \geq 2$  random cuts, with each cut chosen at a random place of the stick. What is the probability that the  $n + 1$  pieces can be put together to form a convex  $(n + 1)$ -gon?

**74. Triads.** Can the plane be tiled with triads, three Jordan arcs meeting in a point? More precisely, a *triad* is a subset of the plane homeomorphic to the union of the three line-segments  $OX$ ,  $OY$  and  $OZ$ , with  $O = (0, 0)$ ,  $X = (1, 0)$ ,  $Y = (0, 1)$  and  $Z = (-1, 0)$ . Then the question is whether the plane can be partitioned into pairwise disjoint triads.



**75. Colouring Complete Graphs.** Suppose that the complete graph  $K_n$ ,  $n \geq 4$ , has a proper edge-colouring with  $n - 1$  colours such that any six edges forming a complete graph on four vertices get either three colours or six. Show that then  $n = 2^m$  for some integer  $m \geq 2$ . Furthermore, for every such  $n$  there is such a colouring.

**76. Symmetric Convex Domains.** Show that every convex planar domain of area 1 contains a centrally symmetric convex domain of area at least  $2/3$ .

**77. Independent Random Variables.** Let  $X_1, X_2, \dots$  be independent  $\pm 1$  random variables with  $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$  for every  $i$ .

- (i) For  $i = 1, 2, \dots$ , set  $Y_i = X_i X_{i+1} X_{i+2}$ . Are the variables  $Y_1, Y_2, \dots$  independent?
- (ii) Let  $\mathcal{S} = \{S_1, S_2, \dots\}$  be a collection of finite sets of natural numbers. For  $i = 1, 2, \dots$ , set  $Z_i = \prod_{j \in S_i} X_j$ . For what families  $\mathcal{S}$  are the random variables  $Z_1, Z_2, \dots$  independent?

**78. Triangles Touching a Triangle.** How many non-overlapping regular triangles can touch a regular triangle if all triangles have the same size?

**79. Even and Odd Graphs.** Call a graph *even* if every vertex has an even degree, and call it *odd* if every vertex has an odd degree. Show that the vertex set of every graph can be partitioned into two sets that induce even subgraphs.

Deduce that there is a partition in which one of the sets induces an even subgraph and the other an odd subgraph.

**80. Packing Squares.** Show that a set of squares of total area 1 can be packed into a square of area 2. Furthermore, show that this is not true if 2 is replaced by a smaller number.

**81. Filling a Matrix.** Consider a  $2k \times 2k$  matrix whose entries are  $1, 2, \dots, 4k^2$  in some order. Show that some row or column contains two entries whose difference is at least  $2k^2$ .

What is the greatest difference one can guarantee?

**82. An Inequality Concerning Triangles.** Show that if  $O$  is a point within a triangle  $ABC$ , and  $P$ ,  $Q$  and  $R$  are the feet of the perpendiculars upon the sides (more precisely, upon the lines determined by the sides) then

$$OA + OB + OC \geq 2(OP + OQ + OR). \quad (1)$$

When does equality hold?

**83. Perfect Difference Sets.** Let  $p \geq 1$  and  $n = p^2 + p + 1$ . A set  $A \subset \mathbb{Z}_n$  is a *perfect difference set* for  $n$  if  $|A| = p + 1$  and for every  $b \in \mathbb{Z}_n$  there are  $a, a' \in A$  such that  $a - a' = b$ . (There are  $p^2 + p$  ordered pairs  $(a, a')$ ,  $a \neq a'$ , so for  $b \neq 0$  there is a unique pair  $(a, a')$  with  $a - a' = b$ .) Equivalently,  $0 \leq a_0 < a_1 < \dots < a_p \leq n$  is a perfect difference set for  $n$  if for every  $b$ ,  $0 < b < n$  there are  $a_i$  and  $a_j$  such that either  $b = a_i - a_j$  or  $b - n = a_i - a_j$ . Note that  $\{0, 1\}$ ,  $\{0, 1, 3\}$  and  $\{0, 1, 3, 9\}$  are perfect difference sets for  $p = 1, 2$  and  $3$ .

Show that if  $p$  is a prime then there is a perfect difference set for  $n = p^2 + p + 1$ .

**84. Difference Bases.** A set  $B$  of integers is a *difference basis* of  $[n]$  if, for every  $d \in [n]$ , the equation  $d = b_i - b_j$  is solvable for some  $b_i, b_j \in B$ . Write  $k(n)$  for the minimal number of elements in a difference basis of  $[n]$ . Show that

$$\lim_{n \rightarrow \infty} k(n)/\sqrt{n} = c$$

for some constant  $c$ ,  $\sqrt{2} \leq c \leq (8/3)^{1/2}$ .

**85. Satisfied Cricketers.** Suppose that a batsman plays, in a given season, a given 'stock' of innings

$$b_1, b_2, \dots, b_n$$

determined in everything except arrangement. Let  $a_i$  be his maximum average for

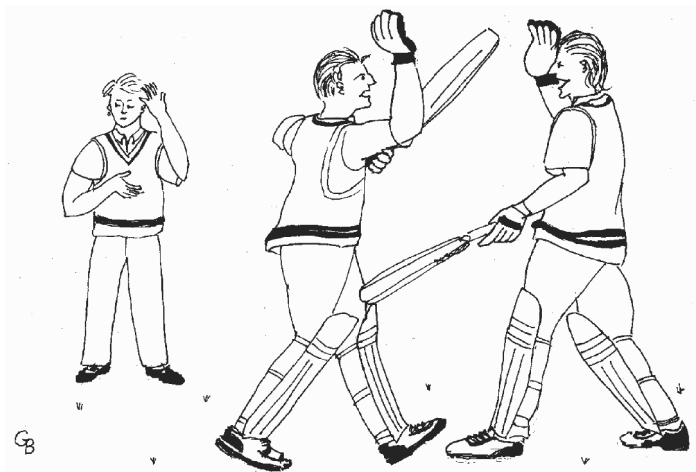


Fig. 13. Two satisfied cricketers.

any consecutive series of innings ending in the  $i$ th, so that

$$a_i = \frac{b_{i^*} + b_{i^*+1} + \cdots + b_i}{i - i^* + 1} = \max_{j \leq i} \frac{b_j + b_{j+1} + \cdots + b_i}{i - j + 1};$$

we may agree that, in case of ambiguity,  $i^*$  is chosen to be as small as possible. Thus, if the innings to date are 82, 4, 133, 0, 43, 58, 65, 53, 86, 30, the batsman says to himself ‘at any rate my average for my last 8 innings is 58.5’ (a not uncommon psychology).

Let  $s(x)$  be a positive function which increases with  $x$ , and let his ‘satisfaction’ after the  $i$ th innings be measured by  $s_i = s(a_i)$ . Finally, let his total satisfaction for the season be measured by

$$S = \sum_{i=1}^n s_i = \sum_{i=1}^n s(a_i).$$

Show that the total satisfaction  $S$  is maximum, for a given stock of innings, when the innings are played in decreasing order.

**86. Random Words.** Consider a random word of length  $2n$ , with  $n$  letters  $A$  and  $n$  letters  $B$ . For  $0 \leq m \leq n$ , let  $p(m, n)$  be the probability that there are precisely  $m$  letters  $B$  in the word that are preceded by at most as many  $A$ s as  $B$ s. What is the value of  $p(m, n)$ ?

**87. Crossing a Chess Board.** Consider an  $n$  by  $m$  chess board (without the colouring) and colour the squares black and white in any way. Show that either a rook can move from the first row to the last using (going over and landing on) only black squares, or a king can move from the first column to the last using only white squares, but not both.

### 88. Powers of Paths and Cycles

- (i) Let  $G = P_\infty^k$  be the  $k$ th power of a 2-way infinite path. For example, let  $V(G) = \mathbb{Z}$  and  $E(G) = \{ij : i < j \leq i + k\}$ . What is the maximal number of edges spanned by  $\ell$  vertices?
- (ii) Let  $G$  be the graph with vertex set  $V(G) = \mathbb{Z}_n$  in which vertex  $i$  is joined to vertex  $j \neq i$  if  $-k \leq i - j \leq k$ , where  $k \geq 1$  and  $n \geq 2k + 1$ . (Thus,  $G = C_n^k$ , i.e.,  $G$  is the  $k$ th power of the  $n$ -cycle  $C_n$ .) Let  $V(G) = A \cup B$  be a partition of the vertex set of  $G$  into sets with at least  $k$  vertices in each. What is the minimal number of edges joining  $A$  to  $B$ ?

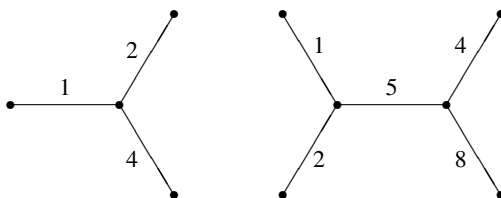


Fig. 14. Two perfect weighted trees.

**89. Powers of Oriented Cycles.** Let  $n \geq 2k + 1 \geq 3$  and let  $\vec{G}$  be the standard orientation of  $C_n^k$ , the  $k$ th power of an  $n$ -cycle  $C_n$ , as in Problem 88. Thus the vertex set of  $\vec{G}$  is the set of integers modulo  $n$ ,  $\mathbb{Z}_n$ , and there is an edge from a vertex  $i$  to a vertex  $j$  if  $j = i + h$  for some  $h$ ,  $1 \leq h \leq k$ .

Note that if  $V(G) = A \cup B$  is a partition of the vertex set  $\mathbb{Z}_n$  into two sets in which  $A$  is a set of  $k$  consecutive vertices, then there are  $k(k + 1)/2$  edges going from a vertex in  $A$  to a vertex in  $B$ . Is this the minimum for every partition  $V(G) = A \cup B$  into two sets with at least  $k$  vertices in each?

**90. Perfect Trees.** An edge-weighted tree is one to whose edges are assigned *weights* (real numbers). The weighted distance  $d_w(x, y)$  between vertices  $x$  and  $y$  is the sum of the weights of the edges on the unique path from  $x$  to  $y$ . A weighted tree is *perfect* if it has  $n$  vertices and the distances between its vertices are  $1, 2, \dots, \binom{n}{2}$ . (Figure 14 shows a perfect tree of order four and one of order six.) Show that if there is a weighted tree of order  $n$  then either  $n$  or  $n - 2$  is a perfect square.

**91. Circular sequences.** Let  $X = (x_1, x_2, \dots, x_n)$  be a circular sequence of integers with sum  $S = \sum_{i=1}^n x_i$ . (A little more formally, the indices are from  $\mathbb{Z}_n$ , the integers modulo  $n$ ; thus  $\dots \equiv x_{-n} \equiv x_0 \equiv x_n \equiv x_{2n} \equiv \dots, \dots \equiv x_{-n+1} \equiv x_1 \equiv x_{n+1} \equiv x_{2n+1} \equiv \dots$ , and so on.) If  $x_k < 0$  then the *reflection of  $X$  in  $k$*  is the circular sequence  $r_k(X) = (x_1, \dots, x_{k-2}, x_{k-1} + x_k, -x_k, x_{k+1} + x_k, x_{k+2}, \dots, x_n)$ :  $x_k$  is added to each of the two neighbouring terms,  $x_{k-1}$  and  $x_{k+1}$ , and  $x_k$  itself is replaced by its modulus  $-x_k$ . In particular, a reflection does not change the sum of the sequence. A sequence is *stable* if it has no negative terms, i.e., has no reflections. Call a sequence  $X_0, X_1, \dots$  a *reflection sequence* if  $X_{i+1}$  is a reflection of  $X_i$  for every  $i$ .

Show that every reflection sequence is finite. Even more, if a reflection sequence starts with  $X$  and ends in the stable sequence  $X_\ell$  then both  $\ell$  and  $X_\ell$  depend only on  $X$  and not on the particular sequence chosen.

**92. Infinite Sets with Integral Distances.** Show that the plane does not contain an infinite set of non-collinear points (i.e., not all on a line) with all pairs of points at integral distances.

**93. Finite Sets with Integral Distances.** Show that for every  $n \geq 3$  there are  $n$  non-collinear points in the plane such that the distance between any two is an integer. Are there also  $n$  points, with no three on a line, such that the distance between any two is an integer?

**94. Cube-free Words.** Is there an infinite word on a two-letter alphabet in which no block occurs three times in succession?

**95. Square-free Words.** Is there an infinite word on a finite alphabet in which no block occurs twice in succession?

In other words, is there a square-free infinite word on a finite alphabet? A *square* is a word of the form  $uu$ , with  $u$  non-empty and, obviously, a word is *square-free* if no block of it is a square.

**96. Addition of Residue Classes**

- (i) Let  $A$  and  $B$  be non-empty subsets of  $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ , the set of congruence classes modulo a prime  $p$ . Show that if  $p \geq |A| + |B| - 1$  then

$$A + B = \{a + b : a \in A, b \in B\}$$

has at least  $|A| + |B| - 1$  elements.

- (ii) For a prime  $p$  and integer  $k$ ,  $1 \leq k \leq p - 1$ , let  $G_{p,k}$  be a circulant digraph (i.e., directed graph) on  $\mathbb{Z}_p$  with out-degree  $k$ . Thus the vertex set of  $G_{p,k}$  is  $\mathbb{Z}_p$ , and there is a  $k$ -subset  $C$  of  $\mathbb{Z}_p \setminus \{0\}$  such that  $\vec{ab}$  is an edge (going from  $a$  to  $b$ ) if and only if  $b - a \in C$ . [The set  $C$  gives the set of *chords*.] Show that every set of  $p - k + 1$  vertices of  $G_{p,k}$  induces a strongly connected subgraph, i.e., if  $W \subset \mathbb{Z}_p$ ,  $|W| = p - k + 1$ , and  $u, v \in W$ , then there are  $w_0 = u, w_2, \dots, w_s = v \in W$  such that  $w_1 - w_0, w_2 - w_1, \dots, w_s - w_{s-1}$  are all in  $C$ .

**97. Sums Congruent to Zero.** Show that every sequence of  $2n - 1$  natural numbers contains  $n$  terms whose sum is divisible by  $n$ . Show also that the assertion is false if  $2n - 1$  is replaced by  $2n - 2$ .

**98. Subwords of Distinct Words.** Let  $u$  and  $u'$  be distinct words of length  $\ell \geq m$ . Show that if  $\ell \leq 2m - 1$  then some word of length  $m$  is a subword of one but not of the other, but this need not hold if  $\ell \geq 2m$ .

**99. Prime Factors of Sums.** Given natural number  $a_1 < a_2 < \cdots < a_n$ ,  $n = 2^k + 1$ , form all  $n(n-1)/2$  two-term sums  $a_i + a_j$ . At least how many prime factors occur in these sums?

### 100. Catalan Numbers

- (i) How many legal bracketings (paranthesizations) are there that use  $n$  pairs of brackets? For example, for  $n = 3$  there are five legal bracketings:  $()()()$ ,  $()(())$ ,  $((())())$ ,  $((())())$  and  $((())())$ .
- (ii) Given  $2n$  points on a circle, in how many ways can one join them with  $n$  non-intersecting chords, with each point on one chord?
- (iii) How many integer sequences  $(a_i)_1^{n-1}$  are there such that  $1 \leq a_1 \leq a_2 \leq \cdots \leq a_{n-1}$  and  $a_i \leq i$  for every  $i$ ?
- (iv) Given  $n \geq 1$ , how many pairs of sequences  $(a_i)_1^k, (b_j)_1^k$  of natural numbers are there such that  $\sum_{i=1}^k a_i = \sum_{j=1}^k b_j = n$  and  $\sum_{j=1}^h b_j \leq \sum_{i=1}^h a_i$  for every  $h$ ,  $h = 1, \dots, k$ , with  $k = 1, \dots, n$ ?
- (v) How many binary plane trees are there with  $n + 1$  end-vertices? A *plane tree* is a rooted tree such that the subtrees at each vertex are linearly ordered; such a tree is *binary* if the root has degree 2 and every other vertex has degree 1 or 3. Figure 15 shows the five binary plane trees with four end-vertices (and so seven vertices altogether).
- (vi) How many unordered pairs of monotone increasing lattice paths are there in  $\mathbb{Z}^2$ , each with  $n + 1$  steps, starting at the origin and ending at the same point, and sharing only the beginning and the end? See Figure 16 for the five possibilities for  $n = 3$ .

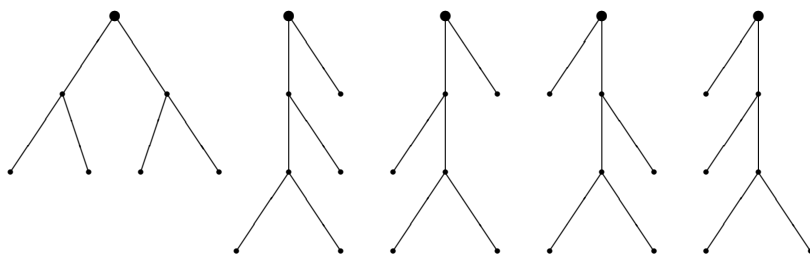


Fig. 15. The five binary plane trees with four end-vertices; the big dots at the top are the roots of the trees.

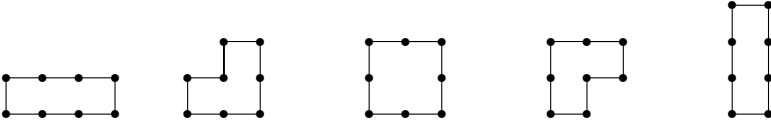


Fig. 16. The five pairs of lattice paths for  $n = 3$  that share only their end-points.

**101. Permutations without Long Decreasing Subsequences.** What is the number of permutations  $a_1 a_2 \dots a_n$  of  $[n]$  that do not contain a decreasing subsequence of length three? Note that for  $n = 3$  there are five such permutations, since the only one ruled out is the permutation 321; for  $n = 4$  there are fourteen: 1234, 1243, 1324, 1342, 1423, 2134, 2143, 2314, 2341, 2413, 3124, 3142, 3412 and 4123.

**102. Random Intervals.** What is the probability that one of  $n \geq 2$  random intervals meets every other interval?

**103. Sums of Convex Bodies**

- (i) Let  $A$  and  $B$  be convex sets of volume  $a^n$  and  $b^n$  in  $\mathbb{R}^n$ . Show that the volume of  $A + B = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in A, \mathbf{b} \in B\}$  is at least  $(a + b)^n$ .
- (ii) Let  $A \subset \mathbb{R}^n$  be a convex set, and put  $h_1 = \inf\{a_n : (a_1, \dots, a_n) \in A\}$  and  $h_2 = \sup\{a_n : (a_1, \dots, a_n) \in A\}$ . Let  $v(t)^{n-1}$  be the  $(n - 1)$ -dimensional volume of  $A \cap H_t$ , where

$$H_t = \{(x_1, \dots, x_{n-1}, t) : x_i \in \mathbb{R} \text{ for } i = 1, \dots, n - 1\}$$

is the hyperplane at height  $t$ . Show that the function  $v(t)$  is concave on the interval  $(h_1, h_2)$ .

**104. Cross-intersecting Families.** A family  $\{(A_i, B_i) : 1 \leq i \leq m\}$  of pairs of finite sets is *cross-intersecting* if  $A_i \cap B_i = \emptyset$  for every  $i$  and  $A_i \cap B_j \neq \emptyset$  for every  $i \neq j$ . Show that if  $\{(A_i, B_i) : 1 \leq i \leq m\}$  is a cross-intersecting family then

$$\sum_{i=1}^m \left( \frac{|A_i| + |B_i|}{|A_i|} \right)^{-1} \leq 1.$$

**105. Saturated Hypergraphs.** An  $r$ -uniform hypergraph of order  $n$  is a pair  $H = (V, E)$ , where  $E$  is a subset of the collection  $V^{(r)}$  of all  $r$ -subsets of a set  $V$  with  $n$  elements. We call  $V$  the set of vertices and  $E$  the set of edges. If  $E = V^{(r)}$  and  $|V| = k$  then  $(H, E)$  is the complete  $r$ -uniform hypergraph with  $k$  vertices.

A hypergraph  $H$  is  $k$ -saturated if it does not contain a  $K_k^{(r)}$ , but the addition of any edge creates a new graph containing a  $K_k^{(r)}$ . At least how many edges are there in a  $k$ -saturated  $r$ -uniform hypergraph?

**106. The Norm of Averages.** Let  $x = (x_k)_1^n$  be a sequence of non-negative reals, and let  $a = a(x) = (a_k)_1^n$  be the sequence of its averages, so that  $a_k = A_k/k$ , where  $A_k = \sum_{i=1}^k x_i$ . Show that for every  $p > 1$  there is a constant  $C_p$  (depending only on  $p$ ) such that

$$\|a\|_p \leq C_p \|x\|_p,$$

where  $\|\cdot\|_p$  is the  $p$ -norm, i.e.,  $\|u\|_p^p = \sum_k |u_k|^p$ .

**107. Carleman's Inequality.** Let  $p > 1$ . Assuming that the inequality in Problem 106 holds with  $C_p = \frac{p}{p-1}$ , show that if  $b_1, \dots, b_n$  are positive numbers then

$$\sum_{k=1}^n (b_1 \cdots b_k)^{1/k} \leq e \sum_{k=1}^n b_k.$$

**108. Triangulating Squares.** For what values of  $n$  is it possible to cut a square into  $n$  triangles of equal area? The second tessellation in Figure 17 shows that every even  $n$  will do.

**109. Strongly Separating Families.** A family  $\mathcal{S} = \{S_1, S_2, \dots, S_t\}$  of subsets of a set  $X$  is said to separate strongly the elements of  $X$  if whenever  $x$  and  $y$  are distinct elements of  $X$ , there is a set  $S_i$  that contains  $x$ , but does not contain  $y$ . What is the minimal number of sets in a family strongly separating the elements of an  $n$ -element set?

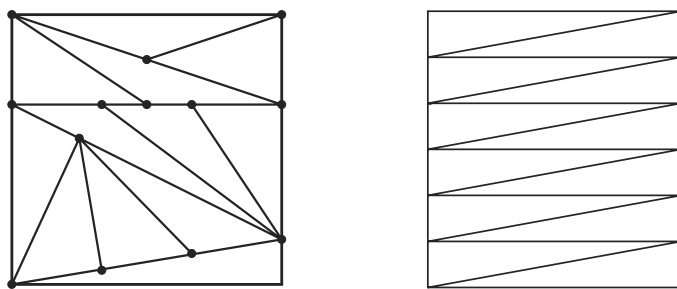


Fig. 17. Tessellations of squares into triangles of the same area.



**110. Strongly Separating Systems of Pairs.** A *strongly separating system* on  $[n]$  is a collection  $(S_1, T_1), \dots, (S_N, T_N)$  of disjoint pairs of subsets of  $[n]$  such that if  $i, j \in [n]$  with  $i \neq j$  then there is a  $k$  with  $i \in S_k$  and  $j \in T_k$ . Writing  $T(n)$  for

$$\min \left\{ \sum_{i=1}^N |S_i \cup T_i| : (S_i, T_i)_{i=1}^N \text{ is a strongly separating system on } [n] \right\},$$

show that if  $n \geq 2$  and  $t$  is the minimal integer such that

$$\binom{t+1}{\lfloor (t+1)/2 \rfloor} > n,$$

then  $T(n) \geq nt$ , with equality if and only if  $n = \binom{t}{\lfloor t/2 \rfloor}$ . In particular,  $T(n) = (\log_2 n + \frac{1}{2} \log_2 \log n + O(1))n$ .

**111. The Maximum Edge-boundary of a Down-set.** Show that the edge-boundary of a down-set in the cube  $Q^n = \{0, 1\}^n$  has at most  $\lceil n/2 \rceil \binom{n}{\lfloor n/2 \rfloor}$  edges. A set  $D \subset Q^n = \{0, 1\}^n$  is a down-set if  $x = (x_i) \in D$ ,  $y = (y_i) \in Q^n = \{0, 1\}^n$ , and  $y_i \leq x_i$  for every  $i$  imply that  $y \in D$ .

**112. Partitioning a Subset of the Cube.** Show that for every subset  $\mathcal{A}$  of the cube  $Q^n$  there are two parallel faces that contain about the same number of elements of  $\mathcal{A}$ . More precisely, for some pair of parallel faces the difference of the numbers of vertices of  $\mathcal{A}$  on these faces is at most

$$\frac{\lceil n/2 \rceil}{n} \binom{n}{\lfloor n/2 \rfloor} \leq 2^n / \sqrt{n}.$$

Show also that if  $\varepsilon > 0$  and  $\eta(n) \rightarrow 0$  then if  $n$  is large enough and  $\mathcal{A} \subset Q^n$  has at least  $2^{(1-\eta(n))n}$  vertices, then for some two parallel faces the difference is at most  $\varepsilon |\mathcal{A}|$ .

**113. Weakly Cross-intersecting Pairs.** Let  $(A_i, B_i)$ ,  $i = 1, \dots, m$ , be pairs of disjoint sets with  $|A_i| = a$  and  $|B_i| = b$  such that  $A_i \cap B_j \neq \emptyset$  for  $1 \leq i < j \leq m$ . Show that  $m \leq \binom{a+b}{a}$ .

**114. Even Sets with Even Intersections.** What is the maximal number of subsets of  $[n] = \{1, \dots, n\}$  such that each set has an even number of elements and any two sets intersect in an even number of elements?

**115. Sets with Even Intersections.** What is the maximal number of subsets of  $[n] = \{1, \dots, n\}$  such that any two of them intersect in an even number of elements?

**116. Even Clubs.** In *Eventown*, with population  $n$ , every club has an even number of members. Strangely, any two clubs have an odd number of members in common. At most how many clubs are there in *Eventown*?

**117. Covering the Sphere.** Show that if the  $n$ -dimensional unit sphere  $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : \|\mathbf{x}\| = 1\}$  is covered by  $n + 1$  sets, each of which is either open or closed, then one of the sets contains a pair of antipodal points, i.e., points  $\mathbf{x}$  and  $-\mathbf{x}$ .

**118. The Kneser Graph.** Let  $n$  and  $k$  be positive integers. Show that if the  $n$ -subsets of  $\{1, 2, \dots, 2n + k\}$  are partitioned into  $k + 1$  subsets  $A_1, \dots, A_{k+1}$  then one of the sets  $A_i$  contains two disjoint  $n$ -sets.

**119. Partitions into Bricks.** Let  $\mathbf{A} = A_1 \times \dots \times A_n$  be the product of non-empty finite sets  $A_1, A_2, \dots, A_n$ . Call a subset  $\mathbf{B}$  of  $\mathbf{A}$  a *brick* if  $\mathbf{B} = B_1 \times \dots \times B_n$ , with each  $B_i$  a non-empty proper subset of  $A_i$ . Show that  $\mathbf{A}$  cannot be partitioned into fewer than  $2^n$  bricks.

**120. Drawing Dense Graphs.** Show that there is a constant  $c > 0$  such that every drawing of a graph with  $n$  vertices and  $m > 5n$  edges contains more than  $c m^3 / n^2$  pairs of crossing edges.

**121. Unit Distances.** Show that there is a constant  $c > 0$  such that  $n$  points in the plane determine at most  $c n^{4/3}$  unit distances.

**122. Point–Line Incidences.** Show that there are at most  $2(2n\ell)^{2/3} + 5n + \ell$  incidences among  $n$  points and  $\ell$  lines in the plane.

**123. Geometric Graphs without Parallel Edges.** A *geometric graph* is a straight-line drawing of a graph in the plane without three collinear vertices. Two edges of a geometric graph are said to be *parallel* if they are the opposite edges of a convex quadrilateral. It is easy to see that for every  $n \geq 4$  there is a geometric graph of order  $n$  and size  $2n - 2$  that contains no pairs of parallel edges (see Figure 18). Show that  $2n - 2$  is the maximum: a geometric graph of order  $n$  with no pairs of parallel edges has at most  $2n - 2$  edges.

Show also that if  $n \geq 4$  is even then there is a geometric graph with  $n$  vertices,  $2n - 2$  edges and minimal degree 3 that does not contain two parallel edges.

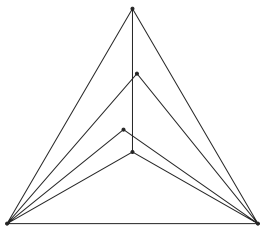


Fig. 18. A geometric graph with 6 vertices and 10 edges that does not contain two parallel edges.

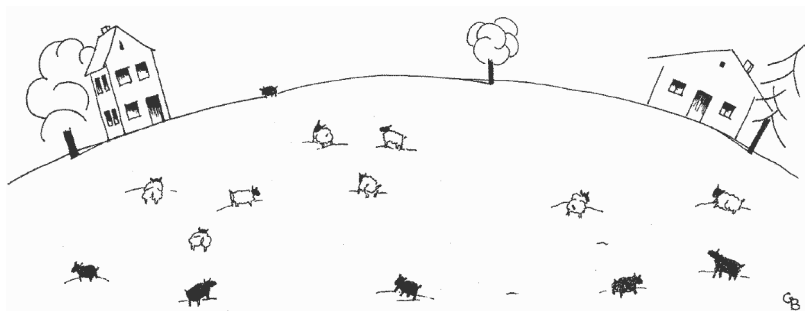


Fig. 19. Black and white sheep in a field.

**124. Shortest Tours.** The *square length*  $s(P)$  of a polygon  $P$  is the sum of the squares of the side-lengths of  $P$ . Given  $n \geq 3$ , determine the smallest number  $s_n$  such that if  $V$  is a set of  $n$  points in a unit square then there is a polygon  $P$  with vertex set  $V$  such that  $s(P) \leq s_n$ .

**125. Density of Integers.** What is the maximal density of integers that are divisible by exactly one member of a finite set of integers greater than 1? To spell it out, given  $2 \leq a_1 < a_2 < \dots < a_k$ , let  $M_n = \{1 \leq b \leq n : b \text{ is divisible by exactly one of the } a_i\}$  and  $M_A = \cup B_n$ . The density of  $M_A$  is  $\delta(M_A) = \lim_{n \rightarrow \infty} |M_n|/n$ . What is the supremum of this density  $\delta(M_A)$ ?

**126. Black and White Sheep.** A field contains a number of stationary black and white sheep. Show that if for every group of at most four sheep, there is a straight line that separates the black sheep from the white, then there is a straight line that separates all the black sheep from the white. What happens in  $n$  dimensions?

**127. Chords of Convex Bodies.** Let  $C$  be a convex body in  $\mathbb{R}^n$ , a compact convex set with non-empty interior. A maximal interval  $[u, v]$  contained in  $C$  is a *chord* of  $C$ . Show that  $C$  contains a point  $c$  that is not far from being central in the following sense: for every chord  $[u, v]$  through  $c$ ,  $\|c - u\|/\|v - u\| \leq n/(n + 1)$ .

**128. Neighbourly Polyhedra.** Can you construct 1000 internally disjoint convex polyhedra in  $\mathbb{R}^3$  any two of which share a face?

**129. Neighbourly Simplices.** Call a family of  $n$ -simplices in  $\mathbb{R}^n$  *neighbourly* if any two of them intersect in an  $(n - 1)$ -dimensional set, i.e., the intersection of any two is a full-dimensional subset of a facet. Show that there are at most  $2^{n+1}$  neighbourly  $n$ -simplices in  $\mathbb{R}^n$ .

**130. The Rank of a Matrix.** For  $i = 0, 1, \dots, n$ , fix an enumeration  $A_1^i, A_2^i, \dots, A_{\binom{n}{i}}^i$  of the  $i$ -subsets of  $[n]$ . Also, for  $1 \leq i \leq j \leq n$ , define an  $\binom{n}{i}$  by  $\binom{n}{j}$  matrix  $N(i, j)$  encoding the containments  $A_s^i \subset A_t^j$ , i.e., let the  $(s, t)$ -entry of  $N(i, j)$  be

$$N(i, j)_{s,t} = \begin{cases} 1 & \text{if } A_s^i \subset A_t^j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus  $N(i, i)$  is an identity matrix,  $N(0, j)$  is an identically 1 row vector and  $N(i, n)$  is an identically 1 column vector. What is the rank of  $N(i, j)$ ?

**131. Modular Intersecting  $k$ -uniform Set Systems.** Let  $p$  be a prime and  $1 < k < n$ ; furthermore, let  $\mu_0, \mu_1, \dots, \mu_r$  be distinct residues modulo  $p$ , with  $k \equiv \mu_0 \pmod{p}$ . Let  $\mathcal{F}$  be a collection of  $k$ -subsets of  $[n]$  such that if  $A$  and  $B$  are distinct elements of  $\mathcal{F}$  then there is an index  $i$ ,  $1 \leq i \leq r$ , such that

$$|A \cap B| \equiv \mu_i \pmod{p},$$

and every  $\mu_i$  occurs in one of these relations. Show that

$$|\mathcal{F}| \leq \binom{n}{r}.$$

**132. Families without Orthogonal Vectors.** Let  $p$  be a prime, and  $X \subset \{1, -1\}^{4p} \subset \mathbb{R}^{4p}$  be a family of balanced vectors no two of which are orthogonal. (Thus every vector in  $X$  has  $2p$  coordinates that are 1 and  $2p$  coordinates that are  $-1$ , and if  $x = (x_1, x_2, \dots, x_{4p})$ ,  $y = (y_1, y_2, \dots, y_{4p}) \in X$  then  $\langle x, y \rangle = \sum x_i y_i \neq 0$ .) Show that

$$|X| \leq 2 \binom{4p-1}{p-1}.$$

**133. Partitioning Sets into Sets of Smaller Diameter.** Let  $k(n)$  be the minimal integer such that if a set  $K \subset \mathbb{R}^n$  has diameter 1 then  $K$  is the union of  $k(n)$  sets of diameter less than 1. Show that  $k(n) \geq c\sqrt{n}$  for some  $c > 1$ .

**134. Periodic Sequences.** Let  $(a_n)_0^\infty, (b_n)_0^\infty$  be periodic sequences of periods  $p$  and  $q$ , respectively. Show that if  $a_n = b_n$  for  $p + q - (p, q)$  consecutive integers  $n$ , then  $a_n = b_n$  for all  $n$ . Here  $(p, q)$  stands for the greatest common divisor of the integers  $p$  and  $q$ .

**135. Periodic Words.** Let  $p < q$  be relatively prime natural numbers, and let  $w$  be a  $p$ -periodic and  $q$ -periodic word of length  $n$ . (Thus  $w = w_1 w_2 \dots w_n$ , with  $w_i + p = w_i$ ,  $w_j + q = w_j$  whenever  $1 \leq i < i + p \leq n$  and  $1 \leq j < j + q \leq n$ .) Show that if  $n \geq p + q - 1$  then  $w$  is constant, i.e.,  $w_1 = w_2 = \dots = w_n$ , but  $w$  need not be constant if  $n < p + q - 1$ .

**136. Points on a Hemisphere: Wendel's Theorem.** Choose  $n$  points at random on the surface of the unit sphere in  $\mathbb{R}^d$  and write  $p_{d,n}$  for the probability that all  $n$  points lie on some hemisphere. Show that

$$p_{d,d+1} = 1 - 2^{-d},$$

and, in general,

$$p_{d,n} = 2^{-n+1} \sum_{k=0}^{d-1} \binom{n-1}{k}.$$

**137. Planar and Spherical Triangles.** Show that if a spherical triangle is reproduced in the plane in such a way that the side-lengths are preserved then each angle of the planar triangle is smaller than the corresponding angle of the spherical triangle.

**138. Hobnails.** The  $n$ -dimensional *hobnail*<sup>†</sup> is the set  $H^n = B^n \cup I \subset \mathbb{R}^{n+1}$ , where

$$B^n = \{x = (x_i)_{i=1}^{n+1} \in \mathbb{R}^{n+1} : \sum_{i=1}^n x_i^2 \leq 1, x_{n+1} = 0\}$$

and

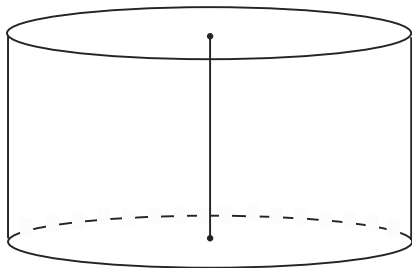
$$I = \{x = (x_i)_{i=1}^{n+1} \in \mathbb{R}^{n+1} : x_1 = \dots = x_n = 0, 0 \leq x_{n+1} \leq 1\}.$$

Thus,  $B^n$ , the *head* of the hobnail  $H^n$ , is the unit ball in  $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ , and  $I$ , the *tang* of  $H^n$ , is the unit interval orthogonal to  $B^n$ , starting at the centre of  $B^n$  (the origin in  $\mathbb{R}^{n+1}$ ), in the positive half-space; see Figure 20.

Show that for every continuous map  $f : H^n \rightarrow \mathbb{R}^n$  there are points  $x, y \in H^n$  at distance 1 such that  $f(x) = f(y)$  and if both  $x$  and  $y$  are in  $B^n$  then the unit segment  $[x, y]$  contains the centre of  $B^n$ .

**139. A Probabilistic Inequality.** Let  $X$  be a random variable with probability density function  $f : \mathbb{R} \rightarrow [0, \infty)$ . Show that for  $p \geq 1$  we have

<sup>†</sup> The OED tells us that a hobnail is a nail with a massive head and short tang, used for protecting the soles of heavy boots and shoes.

Fig. 20. The 2-dimensional hobnail  $H^2$ .

$$\|f\|_\infty \|X\|_p \geq \frac{1}{2}(p+1)^{-1/p},$$

where  $\|\cdot\|_\infty$  and  $\|\cdot\|_p$  are the  $L_p$ -norms, i.e.,  $\|f\|_\infty$  is the infimum of the  $t$  such that the set  $\{x : f(x) > t\}$  has zero measure, and  $\|X\|_p^p = \mathbb{E}(|X|^p) = \int_{-\infty}^{\infty} t^p f(t) dt$ . Show also that equality holds if and only if  $X$  is uniformly distributed on some interval  $[-t, t]$ .

**140. Cube Slicing.** Deduce from the inequality in Problem 139 that for  $n \geq 2$  every section of the solid unit cube  $I^n = \{\mathbf{x} = (x_i)_1^n : -1/2 \leq x_i \leq 1/2 \text{ for every } i\} \subset \mathbb{R}^n$  by an  $(n-1)$ -dimensional subspace  $H$  of  $\mathbb{R}^n$  has volume at least 1. Show also that the volume is 1 if and only if  $H$  is parallel to a face of the cube.

**141. Measures on  $[0, 1]$ .** Let  $\mu_1, \mu_2, \dots, \mu_n$  be non-atomic signed Borel measures on  $[0, 1]$ , so that  $f_j(x) = \mu_j([0, x])$  is a continuous real function on  $[0, 1]$ . Show that  $[0, 1]$  can be cut into two parts with at most  $n$  cuts such that the two parts have equal  $\mu_i$ -measures for each  $i$ . More precisely, for some  $1 \leq r \leq n$  there are points  $\xi_0 = 0 < \xi_1 < \dots < \xi_r < \xi_{r+1} = 1$ , such that with  $A = \bigcup_{i=0}^{\lceil r/2 \rceil} [\xi_{2i}, \xi_{2i+1}]$  and  $B = \bigcup_{i=1}^{\lceil r/2 \rceil} [\xi_{2i-1}, \xi_{2i}] = [0, 1] \setminus A$ , we have

$$\mu_j(A) = \mu_j(B), \quad j = 1, \dots, n.$$

**142. Cutting a Necklace.** Two professors of mathematics steal a valuable necklace made up of beads of different types. They wish to divide the loot fairly but, although they know some mathematics, they have very little idea about the value of the various types of bead. Luckily, the stolen necklace contains an even number of beads of each type: there are  $k$  types of bead and, for  $i = 1, \dots, k$ , there are  $2a_i$  beads of type  $i$ . To achieve a fair and efficient division of their loot, they want to cut the necklace into strands and divide the strands in such a way that each of them gets  $a_i$  beads of type  $i$ . At most how many cuts do they have to make to achieve this?

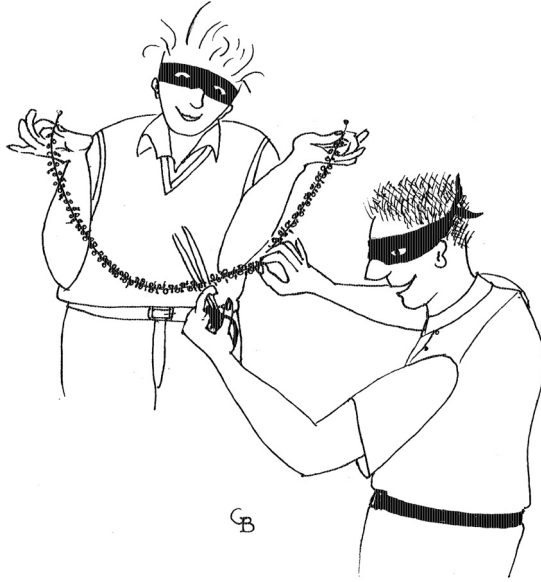


Fig. 21. Thieving mathematicians.

**143. The Norm of an Operator.** Let  $A = (a_{ij})$  be an  $m$  by  $n$  complex matrix, and, for  $x = (x_i)_1^m \in \mathbb{C}^m$  and  $y = (y_i)_1^n \in \mathbb{C}^n$ , set

$$L(x, y) = \sum_{i=1}^m \sum_{j=1}^n a_{ij} x_i y_j.$$

For  $\alpha, \beta > 0$ , let  $M_{\alpha, \beta}$  be the maximum of  $|L(x, y)|$  under the conditions  $\sum_{i=1}^m |x_i|^{1/\alpha} \leq 1$  and  $\sum_{j=1}^n |y_j|^{1/\beta} \leq 1$ . Show that  $\log M_{\alpha, \beta}$  is a convex function of  $\alpha$  and  $\beta$ , i.e., if  $\alpha_0, \alpha_1 = \alpha_0 + \lambda, \beta_0$  and  $\beta_1 = \beta_0 + \mu$  are all greater than 0 then, for  $0 < \sigma < 1$  we have

$$M_{\alpha_\sigma, \beta_\sigma} \leq M_{\alpha_0, \beta_0}^{1-\sigma} M_{\alpha_1, \beta_1}^\sigma,$$

where  $\alpha_\sigma = \alpha_0 + \lambda\sigma$  and  $\beta_1 = \beta_0 + \mu\sigma$ .

**144. Uniform Covers.** A  $k$ -cover of a set  $S$  is a multiset  $\mathcal{C}$  of subsets of  $S$  such that every element of  $S$  is in exactly  $k$  members of  $\mathcal{C}$ . For example,

$$\mathcal{C} = \{\{1, 2\}, \{1, 2\}, \{1, 3, 4\}, \{2, 3, 4\}, \{3, 4\}\}$$

is a 3-cover of  $S = \{1, 2, 3, 4\}$ . A *uniform cover* is a  $k$ -cover for some  $k \geq 1$ . A uniform cover which is not the disjoint union of two uniform covers of  $S$  is

*irreducible*; equivalently, a uniform cover is irreducible if it has no proper uniform subcover. The 3-cover  $\mathcal{C}$  above is clearly irreducible.

Show that every finite set has only finitely many irreducible uniform covers.

**145. Projections of Bodies.** A *body* in  $\mathbb{R}^n$  is a compact set which is the closure of its interior. Given a body  $K \subset \mathbb{R}^n$  and a subset  $A$  of  $[n] = \{1, \dots, n\}$ , denote by  $K_A$  the orthogonal projection of  $K$  onto the subspace of  $\mathbb{R}^n$  spanned by  $\{e_i : i \in A\}$ , where  $(e_1, \dots, e_n)$  is the standard basis for  $\mathbb{R}^n$ . Write  $|K_A|$  for the  $|A|$ -dimensional volume of  $K_A$ .

Show that if  $K$  is a body in  $\mathbb{R}^n$  and  $\mathcal{C}$  is a  $k$ -cover of  $[n]$  (see Problem 144) then

$$|K|^k \leq \prod_{A \in \mathcal{C}} |K_A|.$$

**146. Bodies and Boxes.** A *box* in  $\mathbb{R}^n$  is a rectangular parallelepiped whose sides are parallel to the coordinate axes. Using the notation and terminology of Problem 145, show that for every body  $K$  in  $\mathbb{R}^n$  there is a box  $B$  such that  $|B| = |K|$  and  $|B_A| \leq |K_A|$  for every  $A \subseteq [n]$ .

**147. Intersecting Uniform Set Systems.** A set system (or *hypergraph*)  $\mathcal{F}$  is said to be  $k$ -uniform if  $|A| = k$  for every  $A \in \mathcal{F}$ , and it is *strictly  $S$ -intersecting* if

$$S = \{|A \cap B| : A, B \in \mathcal{F}, A \neq B\}.$$

Show that if  $\mathcal{F}$  is a  $k$ -uniform, strictly  $S$ -intersecting family of subsets of  $[n] = \{1, \dots, n\}$ , then  $|\mathcal{F}| \leq \binom{n}{s}$ , where  $s = |S|$ .

**148. Intersecting Set Systems.** For  $S \subset [n] = \{1, \dots, n\}$ , a collection of sets is said to be  $S$ -intersecting if  $|A \cap B| \in S$  whenever  $A$  and  $B$  are distinct members of it. Let  $\mathcal{F}$  be an  $S$ -intersecting family of subsets of  $[n]$ . Show that

$$|\mathcal{F}| \leq \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{s}, \quad (2)$$

where  $s = |S|$ .

**149. Maps from  $S^n$ .** For what values of  $d$ ,  $0 < d \leq 2$ , does every continuous map from the unit sphere  $S^n \subset \mathbb{R}^{n+1}$  into  $\mathbb{R}^n$  send two points at Euclidean distance (precisely)  $d$  into the same point? Borsuk's theorem (assertion (2) of Problem 41) tells us that  $d = 2$  is such a value.

**150. Closed Covers of  $S^n$ .** Let  $0 < d \leq 2$ , and let  $F_1, \dots, F_{n+1}$  be closed sets covering the  $n$ -dimensional unit sphere  $S^n$ . Is there an index  $i$ ,  $1 \leq i \leq n+1$ , such that  $F_i$  contains two points at distance  $d$ ?



**151. Spherical Pairs.** Let  $X$  be a compact metric space, and  $\varphi : X \rightarrow X$  a distance-preserving involution such that  $d(x, \varphi(x)) = \text{diam}(X)$  for every  $x \in X$ , where  $\text{diam}(X) = \max\{d(y, z) : y, z \in X\}$  is the diameter of  $X$ . We say that  $(X, \varphi)$  is an  $n$ -spherical pair if for every continuous map  $f : X \rightarrow \mathbb{R}^n$  there exists a point  $x \in X$  such that  $f(x) = f(\varphi(x))$ .

Show that  $(X, \varphi)$  is a 1-spherical pair if and only if  $X$  contains a component  $C$  such that  $C = \varphi(C)$ . Show also that if  $(X, \varphi)$  is non-spherical then  $X$  can be partitioned into disjoint compact sets  $W_1$  and  $W_2$  such that  $\varphi(W_1) = W_2$ .

**152. Realizing Distances.** A metric space  $X$  is said to realize a distance  $d$  if there are points  $x, y \in X$  such that  $d(x, y) = d$ . Also, we say that  $X$  realizes all distances if it realizes every  $d$  such that  $0 < d < \text{diam}(X) = \sup\{d(x, y) : x, y \in X\}$ .

Let  $(X, \varphi)$  be an  $n$ -spherical pair, as in Problem 151. As usual, a cover  $X = \bigcup_{i=1}^m F_i$  is said to be *closed* if each  $F_i$  is closed. Show that

(i) for every closed cover  $X = \bigcup_{i=1}^{n+1} F_i$  and every  $0 < \alpha \leq \text{diam}(X)$  there exists  $j \in [n+1]$  such that  $F_j$  realizes  $\alpha$ ;

(ii) for every closed cover  $X = \bigcup_{i=1}^n F_i$ , there exists an index  $j \in [n]$  such that  $F_j$  realizes all distances  $0 < \alpha \leq 2$ .

**153. A Closed Cover of  $S^2$ .** Let  $f : S^2 \rightarrow S^2 \subset \mathbb{R}^3$  be a continuous map, and let  $S^2 = F_1 \cup F_2 \cup F_3$ , where the sets  $F_i$  are closed. Show that for some  $x \in S^2$ , the points  $x$  and  $f(x)$  belong to the same set  $F_i$ .

**154. A Friendly Party.** Suppose that in a party of  $n \geq 3$  people, every two people have exactly one common friend. Show that  $n$  is odd, one of the people is a friend of everybody, and everybody else has precisely two friends.

**155. Polarities in Projective Planes.** A polarity in a finite projective plane is a 1-1 correspondence between points and lines that preserves incidences. Show that every polarity has a fixed point, i.e., a point which is mapped into a line incident with the point itself.

**156. Permutations of Vectors.** Let  $x_1, \dots, x_n$  be vectors of norm at most 1 in a  $d$ -dimensional normed space such that  $x_1 + \dots + x_n = 0$ . Show that there is a permutation  $\pi \in S_n$  such that  $\|x_{\pi(1)} + \dots + x_{\pi(k)}\| \leq d$  for every  $k$ .

**157. An American Story.** In a prison in the USA, there are twenty prisoners on Death Row. To amuse himself and to taunt the inmates, the Governor of the prison announces that they can escape execution if only they pass a little test. Their names are to be placed in twenty identical boxes, one in each; then the boxes are to be closed and arranged in a row in a closed room. The prisoners are to be led into the

room one by one, and are allowed to open twelve boxes to search for their names. Every prisoner will have to leave the room as he found it and, having left it, is not allowed any communication with the others. However, the prisoners are allowed to agree on a strategy *before* the first one is led into the room. The test is for every one of them to find his name.

Thus, the Governor promises that if *every* prisoner finds his name then they will all be reprieved; however, if *even one* of them fails to find his own name then all of them will be executed. The shrewd Governor knows full well that if the prisoners choose the boxes at random (and what else could they do?) then their chances of reprieve is only  $(3/5)^{20} < 0.00004$ . Can the prisoners do any better?

**158. The Angel and the Devil.** The Angel (of speed  $c$ ) and the Devil play the following deadly game on the infinite two-dimensional lattice,  $\mathbb{Z}^2$ , considered as an infinite chessboard with uncoloured squares. The two players alternate moves. Initially, at time  $t = 0$ , the Angel is on some square  $a_0$ . At time  $t = 1$ , the Devil comes along and eats a square  $d_1$  different from  $a_0$ , which means that the Angel can no longer land on it at any future time. On her turn, the Angel flies to a square  $a_1$  different from  $d_1$  and within  $\ell_\infty$  distance  $c$  of her current square,  $a_0$ , i.e., within  $c$  king's moves from  $a_0$ . This procedure is repeated at every time step.

Thus, at time  $t$ , when the Angel is on square  $a_{t-1}$ , the Devil eats a square  $d_t$  other than  $a_{t-1}$ , and the Angel flies to a square  $a_t$  which has not been eaten by the Devil, i.e., to a square other than  $d_1, d_2, \dots, d_t$ , and is within distance  $c$  of  $a_{t-1}$ . Denoting a square by the coordinates of its bottom left vertex, what we demand is that  $a_{t-1} = (x_{t-1}, y_{t-1})$  and  $a_t = (x_t, y_t)$  satisfy  $d(a_{t-1}, a_t) \equiv \max\{|x_t - x_{t-1}|, |y_t - y_{t-1}|\} \leq c$ .

The game is won by the Devil if she can trap the Angel in a bounded region forever, and the Angel wins if she can get arbitrarily far from the origin. Both playing optimally, who wins the Angel and the Devil game?

**159. The Point-Line Game.** *Point* and *Line* play the following game. They take turns to move, with *Point* starting and picking a point  $x_0$  of the unit disc  $D$ . On its turn, *Line* draws a line  $\ell_0$  through  $x_0$ , to which *Point* responds by picking a point  $x_1$  of  $D$  on the line  $\ell_0$ . Then *Line* draws a line  $\ell_1$  through  $x_1$ , *Point* picks a point in  $D$  on the line  $\ell_1$ , and so on. In this way, *Point* and *Line* construct a sequence of points  $x_0, x_1, \dots$  in the unit disc  $D$ , together with lines  $\ell_0, \ell_1, \dots$ , with  $\ell_n$  incident with the points  $x_n$  and  $x_{n+1}$ . *Line* wins if the sequence  $(x_n)$  is convergent, otherwise the game is a win for *Point*.

Who has a winning strategy for this Point-Line Game?



## 2

### The Hints

1. Let  $O$  be the centre of the circle,  $L$  the lion and  $M$  the Christian. What happens if  $L$  keeps on the radius  $OM$  and approaches  $M$  at top speed?
3. What about a random rotation?  
For the second part, use a suitable approximation.
4. Can the line be partitioned into countably many closed sets?
5. Try local changes.
7. Imagine that you can use negative amounts of fuel as well (i.e., can get a loan for your future intake), and every time you get to a town, you get a new lot of fuel. What happens if you go round and round the circuit?
10. Ask the same question for a ‘suitable’ countable set.
11. Which sums seem to be most likely to come up, and which ones seem least likely?
14. Show that the following greedy algorithm constructs a sufficiently large independent set. Pick a vertex  $x_1$  of minimal degree in  $G_1 = G$ , and let  $G_2$  be the graph obtained from  $G_1$  by deleting  $x_1$  and its neighbours. Pick a vertex  $x_2$  of minimal degree in  $G_2$ , and let  $G_3$  be obtained from  $G_2$  by removing  $x_2$  and its neighbours. Proceed in this way, stopping with  $G_\ell$  and  $x_\ell$ , when  $G_\ell$  is a complete graph. By construction, the set  $\{x_1, \dots, x_\ell\}$  is independent. Show that  $\ell \geq i(G)$ , where

$$i(G) = \sum_{v \in V} \frac{1}{d(v) + 1}.$$

16. Let them play 21 games and then decide the winner.
21. Place  $1, 2, \dots$  into the matrix one by one, and stop when you get to an appropriate place.
22. This is an easy consequence of the Baire category theorem.

**24.** This is a great example of a ‘proof by pictures’. Alternatively, apply local moves to change a tiling into a canonical tiling.

**27.** Suppose that  $K_n = \bigcup_{i=1}^k B_i$ , where  $B_1, \dots, B_k$  are bipartite graphs, and  $B_i$  has vertex classes  $U_0^{(i)}$  and  $U_1^{(i)}$ , say. Let  $\varepsilon_i$  be 0 or 1 for every  $i$ ,  $i = 1, \dots, k$ . What can you say about  $V(K_n) \setminus \bigcup_{i=1}^k U_{\varepsilon_i}^{(i)}$ ?

**30.** Apply Helly’s theorem from Problem 29.

**34.** Find a simple numerical invariant of the set of infected sites that does not increase as the infection spreads.

**35.** Suppose a set  $S$  of infected sites is such that it cannot infect any new sites. What does  $S$  look like?

**36.** Prove that  $i = 1$  will do.

**44.** The trisectors divide the triangle into seven small triangles. Experiment with some examples (or simply do some computations) to find out the angles of all these triangles in terms of the angles of the original triangle *assuming that the small triangle in the middle is indeed equilateral*. Once the angles are guessed, there are several obvious ways to attack the problem.

**45.** A *rooted graph*  $G^*$  is a graph  $G$  with a specified vertex, its *root*. Let us write  $N(G^*; n)$  for the number of subtrees of  $G$  with  $n + 1$  vertices, one of which is the root. Show that if  $G$  has maximal degree  $d \geq 2$  then  $N(G^*; n) \leq N(T_d^*; n) = N(T_{d,n}^*; n)$ , where  $T_d$  is the (infinite) *Cayley tree* of degree  $d$ , i.e., the infinite tree in which each vertex has degree  $d$ , and  $T_{d,h}^*$  is the rooted tree of height  $h$  in which every vertex at distance less than  $h$  from the root has degree  $d$ , and every vertex at distance  $h$  from the root is a leaf (has degree 1). Note that  $N(T_d^*; n) = N(T_{d,h}^*; n)$  for every  $h \geq n$ .

**46.** Use generating functions.

**47.** Let  $S \subset \mathbb{R}^n$  be a set of points such that  $\|\mathbf{s} - \mathbf{s}'\| = \alpha$  or  $\beta$ , whenever  $\mathbf{s}, \mathbf{s}' \in S$ ,  $\mathbf{s} \neq \mathbf{s}'$ . For  $\mathbf{s} \in S$  define a function  $f_{\mathbf{s}} : S \rightarrow \mathbb{R}$  by

$$f_{\mathbf{s}}(\mathbf{x}) = (\|\mathbf{x} - \mathbf{s}\|^2 - \alpha^2)(\|\mathbf{x} - \mathbf{s}\|^2 - \beta^2),$$

where  $\|\mathbf{s} - \mathbf{t}\|$  is the distance between  $\mathbf{s}$  and  $\mathbf{t}$ . What can you say about the set of functions

$$\mathcal{F}_S = \{f_{\mathbf{s}} : \mathbf{s} \in S\}?$$

**48.** Writing  $f(n)$  for the minimum in question, it is trivial to show that  $f(1) = 0$ ,  $f(2) = 1$ ,  $f(3) = 3$  and  $f(4) = 4$ , and it is also very easy to see that  $f(n) \leq 2n - 4$ . The real problem is to prove that  $2n - 4$  calls are needed for  $n \geq 4$ . Supposing that this is not the case, let  $n \geq 5$  be the minimal  $n$  for which there is a

suitable system  $S$  of  $2n - 5$  calls. Show that in this system no don hears his own information in any call. Use this information to arrive at a contradiction.

**49.** For simplicity, we call the elements  $p_i$  *points* and the sets  $L_j$  *lines*, so we have the trivial condition that no line contains all the points and the real condition that any two points are contained in exactly one line. We have to conclude that there are at least as many lines as points. Note that we may assume that every line contains at least two points.

Denote by  $d_i$  the number of lines through the point  $p_i$ , and by  $D_j$  the number of points contained in the line  $L_j$ . Show that if a point  $p_i$  is not contained in a line  $L_j$  then  $d_i \geq D_j$ .

**50.** Let  $A = (a_{ij})$  be the incidence matrix of the set system  $\mathcal{E}$ :  $a_{ij}$  is 1 if  $i \in E_j$  and 0 otherwise, and let  $B = (b_{ij})$  be the adjacency matrix of  $\mathcal{E}$ :  $b_{ij} = |E_i \cap E_j|$ . Thus  $A$  is an  $m \times n$  matrix and  $B$  is an  $n \times n$  matrix, each with non-negative integer entries. What is the rank of  $B$ ?

**56.** Assuming that  $0 < a < b < c$  and  $b = qa + r$ , with  $0 \leq r < a$ , give a sequence of legal moves at the end of which  $b$  goes down to  $r$ .

**57.** (i) Consider the left-most of the points.

**58.** Use the  $n$ th Chebyshev polynomial in the form  $T_n(x) = 2 \cos(n \cos^{-1}(x/2))$ . Check that  $T_n$  is a monic polynomial of degree  $n$  with real coefficients.

**59.** Use vectors in  $\mathbb{Z}_2^n$ , the vector space over the field  $\mathbb{Z}_2$  of two elements, 0 and 1, with  $1 + 1 = 0$ , to represent the clubs.

**60.** This is a joke!

**68.** Apply the result in Problem 67.

**70.** Enlarge the boxes.

**71.** Endow  $X$  with a cyclic order, and give an upper bound for the number of  $k$ -intervals in  $\mathcal{A}$ , i.e., the number of sets  $k$ -sets whose elements are consecutive in the cyclic order.

**72.** For  $A \in \mathcal{A}$ , consider the set  $S_A$  of permutations  $p_1 p_2 \dots p_n$  of  $[n]$  such that  $A = \{p_1, p_2, \dots, p_a\}$ , where  $a = |A|$ .

**73.** Show more, namely that if  $P$  is any one of the five points then there is a closed half-space containing four of the points, with  $P$  on the boundary of the half-space.

**74.** Show that the plane does not contain an uncountable family of pairwise disjoint triads.

**79.** Prove the first assertion by induction on the order of the graph. To prove the induction step, if our graph is not even, delete a vertex of odd degree and toggle the pairs of its neighbours.

**81.** At least how many entries share a row or column with one of the entries  $1, 2, \dots, k^2$ ?

**85.** Let  $\mathbf{b} = (b_i)_1^n$  be a real sequence and let  $\mathbf{b}^* = (b_i^*)_1^n$  be the same sequence arranged in descending order, with the obvious treatment of repetitions. Let  $(a_i)$  and  $(a_i^*)$  be the derived sequences of ‘maximal averages’. Note that the assertion in the question follows if we show that, for every  $x$ , the number of  $i$  with  $a_i \geq x$  is at most the number of  $i$  with  $a_i^* \geq x$ .

**87.** Replace the squares by octagons.

**88.** To do part (ii), reformulate the problem, and apply induction on an unlikely parameter.

**89.** Recall the solution of Problem 88.

**90.** Classify the vertices according to the parity of their distance from a fixed ‘root’.

**91.** ‘Lift’ a circular sequence to a doubly infinite sequence.

**92.** Prove the following stronger statement. If  $A, B, C$  are three points not on a line, and  $k = \lfloor \max(AB, BC) \rfloor$ , i.e.,  $k$  is the integer part of the longer of the two distances  $AB$  and  $BC$ , then there are at most  $4(k+1)^2$  points  $P$  such that  $PA - PB$  and  $PB - PC$  are integral, where  $XY$  denotes the distance between points  $X$  and  $Y$ .

**94.** Construct arbitrarily long words by the ‘substitution method’: find words  $w_0 = 0 \dots 0$  and  $w_1 = 1 \dots 1$  on the alphabet  $\{0, 1\}$  such that if  $u_0 = 0$  and for  $k \geq 0$  we obtain  $u_{k+1} = u_k[w_0, w_1]$  from  $u_k$  by replacing each occurrence of  $i$  by  $w_i$ , then none of  $u_1, u_2, \dots$  contains three identical consecutive blocks.

**95.** Construct words  $u_0, u_1, \dots$  on the alphabet  $\{1, 2, 3\}$  by the ‘substitution method’: starting with  $u_0 = 1$ , define  $u_{n+1}$  from  $u_n$  by replacing each occurrence of  $i$  by  $i(i+1)(i+2)$  if it is in an odd place, and by  $(i+2)(i+1)i$  if it is in an even place, with addition defined modulo 3.

**97.** Show that the assertion holds when  $n$  is a prime, and then prove it for all  $n$ .

**99.** For a prime  $p$  and integer  $a$ , write  $v_p(a)$  for the exponent of  $p$  in the prime factorization of  $a$ , i.e., for maximal exponent  $k$  with  $p^k | a$ . Call a set  $S$  of natural numbers  $p$ -simple if  $v_p(a+b) = \min\{v_p(a), v_p(b)\}$  for all  $a, b \in S, a \neq b$ . Show that every finite set  $A \subset \mathbb{N}$  has a  $p$ -simple subset  $S$  with  $|S| \geq |A|/2$ .

**101.** Establish a one-to-one correspondence with monotone increasing lattice paths from  $(0, 0)$  to  $(n, n)$  that do not rise above  $y = x$ .

**102.** To construct our random intervals, proceed as follows. Throw  $2n+1$  points at random on the (oriented) circle:  $2n$  of these points come in pairs (so they are indistinguishable objects), and the last point,  $X$ , is a marker for the start and finish of our interval. Clearly, our random intervals can be recovered from these points: we start at  $X$  and reading around the circle we use each pair to give us an interval. The distribution of the intervals is exactly as required.

**103.** It suffices to prove (i) in the case when  $A$  and  $B$  are unions of finitely many boxes, where a *box* is a set of the form  $\prod_{i=1}^n [c_i, c'_i] = \{\mathbf{x} = (x_i)_1^n : c_i \leq x_i \leq c'_i, i = 1, \dots, n\}$ .

**105.** This is immediate from Bollobás's Lemma in Problem 104.

**106.** Assuming, as we may, that the  $x_k$  are positive reals, rewrite the bound  $\sum_k a_k^p = \sum_k (A_k/k)^p$  using telescopic sums.

**108.** A square cannot be cut into an odd number of triangles of equal area. Rather surprisingly, this can be shown with the aid of 2-adic norms.

**109.** The answer is the minimal  $t$  such that

$$n \leq \binom{t}{\lfloor t/2 \rfloor}.$$

In particular, the minimum is about  $\log_2 n$ .

**110.** Recall from Problem 104 that a collection  $\{(A_j, B_j) : 1 \leq j \leq n\}$  of pairs of sets is *cross-intersecting* if  $A_j \cap B_j = \emptyset$  for every  $j$  and  $A_j \cap B_h \neq \emptyset$  whenever  $j \neq h$ .

Show that  $(S_1, T_1), \dots, (S_N, T_N)$  is a strongly separating system on  $[n]$  if and only if  $\{(A_j, B_j) : 1 \leq j \leq n\}$  is a cross-intersecting family, where

$$\begin{aligned} A_j &= \{i : j \in S_i\}, \\ B_j &= \{i : j \in T_i\}, \end{aligned}$$

and deduce the required bound from Bollobás's Lemma in Problem 104.

**112.** Show that we may suppose that  $\mathcal{A}$  is a down-set.

**113.** Embed  $X = \bigcup_{i=1}^m (A_i \cup B_i)$  into  $V = \mathbb{R}^{a+1}$  in general position, and let  $u_i$  be a normal of the subspace spanned by the elements of  $A_i$ . For each  $i$ , use the set  $B_i$  to define symmetric  $b$ -linear functional  $f_{B_i}$  on  $V$ . As the dimension of the space of  $b$ -linear functionals on  $V$  is  $\binom{a+b}{a}$ , it suffices to prove that the functionals  $f_{B_i}$  are independent.

**114.** Again, represent each set  $X$  by its characteristic vector  $\mathbf{x} \in \mathbb{Z}_2^n$ , as in Problem 59, and endow  $V = \mathbb{Z}_2^n$  with the inner product  $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i$ , where  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and  $\mathbf{y} = (y_1, y_2, \dots, y_n)$ .

**115.** Split the collection into even and odd sets. Thus, let  $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$  be a collection of subsets of  $[n]$  such that any two of them intersect in an even number of elements, with every element of  $\mathcal{A}_0$  even (i.e., with an even number of elements) and every element of  $\mathcal{A}_1$  odd. Assuming that  $|\mathcal{A}_1| = \ell$ , say, enlarge the ground set  $[n]$  with  $\ell$  new elements, one for each odd set, so that the new system consists of even sets, with any two of them meeting in an even number of elements. Enlarge this system and apply the result in Problem 114 to bound the number of sets in  $\mathcal{A}$ .



**116.** For  $n$  odd, apply the result proved in Problem 59. For  $n$  even, try to apply the inequality proved in the solution of Problem 115 that if there is a collection of  $k$  even and  $\ell$  odd subsets of  $[n]$  such that any two subsets intersect in an even number of elements then  $k \leq 2^{\lfloor (n-\ell)/2 \rfloor}$ .

**118.** Place  $2n + k$  points on the unit sphere  $S^{k+1} = \{\mathbf{x} \in \mathbb{R}^{k+2} : \|\mathbf{x}\| = 1\}$  in general position, and let  $A_1, \dots, A_{k+1}$  be a partition of the  $n$ -subsets of these points. Use these  $A_i$  to define  $k + 1$  open subsets of  $S^{k+1}$ , and extend it with the complement of the union to obtain a cover of  $S^{k+1}$  with  $k + 2$  sets,  $k + 1$  of which are open and one is closed. Apply the extension of the Lusternik–Schnirelmann–Borsuk theorem from Problem 117.

**119.** For each  $i$ ,  $1 \leq i \leq n$ , let  $R_i$  be a subset of  $A_i$  chosen uniformly at random from among the *odd* subsets of  $A_i$ , and set  $\mathbf{R} = R_1 \times \dots \times R_n$ . Thus  $\mathbf{R}$  is a random *odd* product set: it is obtained by choosing it at random from among all the product subsets of  $\mathbf{A}$  with an odd number of elements. Equivalently, construct  $\mathbf{R}$  as follows: pick an odd subset  $R_i$  of  $A_i$  at random (so that each odd subset is taken with the same probability  $2^{-|A_i|+1}$ ) and set  $\mathbf{R} = R_1 \times \dots \times R_n$ . Consider the parity of the intersection of this random subset  $\mathbf{R}$  with the bricks in a partition.

**120.** Consider a plane drawing of the graph with the minimal number of pairs of crossing edges, and take a random subgraph by selecting the vertices with a suitable probability  $p$ , independently of each other.

**121.** Given  $n$  points in the plane, draw circles of unit radius about these points, and use the *arcs* of these circles to define a multigraph on these points. Apply the crossing number result of Leighton, Ajtai, Chvátal, Newborn and Szemerédi proved in Problem 120.

**122.** Use segments of the lines to define a graph on the point set, and apply the theorem of Leighton, Ajtai, Chvátal, Newborn and Szemerédi in Problem 120.

**124.** Find a shortest (in the square sense) polygonal path between the two vertices of the hypotenuse of an isosceles right-angled triangle with unit side-lengths with  $n$  given internal vertices.

**128.** Choose points  $P_1, P_2, \dots$  on the moment curve  $(x, x^2, x^3)$  and show that their Voronoi cells are such that any two of them share a face. Thus there is a partition of  $\mathbb{R}^3$  into possibly infinite polyhedra, any two of which share a face. The *Voronoi cell*  $C_i$  of a point  $P_i$  is the set of points that are at most as far from  $P_i$  as from any other point  $P_j$ , i.e.,

$$C_i = \{P \in \mathbb{R}^3 : |P - P_i| \leq |P - P_j| \text{ for all } j \neq i\}.$$

**129.** Let  $S_1, \dots, S_p$  be neighbourly  $n$ -simplices in  $\mathbb{R}^n$ , and let  $H_1, \dots, H_q$  be the all the hyperplanes determined by the facets of these simplices. Each hyperplane

$H_j$  determines two closed half-spaces  $H_j^+$  and  $H_j^-$  of  $\mathbb{R}^n$ . (Here the signs have no significance whatsoever: they are used to distinguish the two half-spaces.) Define a  $p$  by  $q$  matrix  $B = (b_{ij})$  by setting

$$b_{ij} = \begin{cases} 0, & \text{if } S_i \text{ has no facets on } H_j, \\ 1, & \text{if } S_i \text{ has a facet on } H_j \text{ and } S_i \subset H_j^+, \\ -1, & \text{if } S_i \text{ has a facet on } H_j \text{ and } S_i \subset H_j^-. \end{cases}$$

Use this matrix to prove that  $p \leq 2^{n+1}$ .

**131.** Make use of the matrices  $N(i, j)$  studied in Problem 130.

**132.** For  $n = 4p - 1$  and  $k = 2p - 1$  find a family  $\mathcal{F}_0$  of  $k$ -subsets of  $[n]$  such that  $|\mathcal{F}_0| \geq |X|/2$  and if  $A, B \in \mathcal{F}_0$  then  $|A \cap B| \neq p - 1$ . Apply the method of Problem 131 to bound  $|\mathcal{F}_0|$ .

**133.** An  $(n, k, t)$ -family is a  $t$ -intersecting  $k$ -uniform hypergraph  $(\mathcal{F}, V)$  with  $n$  vertices. Let  $h_{k,t}(n)$  be the minimal integer such that every  $(n, k, t)$ -family is the union of  $h_{k,t}(n)$   $(n, k, t + 1)$ -families, and set  $h(n) = \max_{k,t} h_{k,t}(n)$ . Check that  $h(n) \leq k(n)$ .

Let  $p$  be a prime and set  $W = [4p]^{(2)}$ , i.e., let  $W$  be the edge set of the complete graph on  $[4p]$ . Let  $\mathcal{F}$  be the collection of  $4p^2$ -subsets of  $W$  consisting of the edge sets of complete bipartite graphs on  $[4p]$ , with  $2p$  vertices in each class. Use the Frankl–Wilson theorem in Problem 132 to show that  $\mathcal{F}$  gives an example showing that

$$h(2p(4p - 1)) \geq (1.2)^p.$$

**134.** First, note that we may assume that  $a_n = b_n$  for  $n = 0, 1, \dots, p + q - (p, q)$ . Note also that we may assume that the  $a_n$  and  $b_n$  are real numbers, say, and define formal power series by setting  $F(X) = \sum_{n=0}^{\infty} a_n X^n$  and  $G(X) = \sum_{n=0}^{\infty} b_n X^n$ . Then our aim is to prove that  $F(X) = G(X)$ .

**136.** Show that

$$p_{d,n} = \frac{1}{2}(p_{d,n-1} + p_{d-1,n-1}).$$

**138.** Apply Borsuk's theorem from Problem 41.

**141.** Apply Borsuk's theorem in the form of assertion (3) in Problem 41.

**142.** Apply the Hobby–Rice theorem from Problem 141.

**147.** (i) Let  $x_1, x_2, \dots, x_n$  be independent variables, and for  $I \subset [n]$  set  $x_I = \prod_{i \in I} x_i$ . Note that  $x_{\emptyset} = 1$  and  $x_{[n]} = x_1 \cdots x_n$ . Let  $\mathcal{P}(n) = \mathcal{P}([n]) \cong \mathcal{Q}_n$  be the power set of  $[n]$ , the set of all  $2^n$  subsets of  $[n]$ , which is trivially identified with the  $n$ -dimensional discrete cube  $\mathcal{Q}_n = \{0, 1\}^n \subset \mathbb{R}^n$ . Show that if  $f : \mathcal{P}(n) \rightarrow \mathbb{R}$

is such that  $f(I) \neq 0$  whenever  $|I| \leq r$  then the set  $\{x_I f : |I| \leq r\}$  of functions on  $\mathcal{P}(n)$  is linearly independent.

(ii) Let  $S = \{r_1, r_2, \dots, r_s\}$  and  $\mathcal{F} = \{A_1, A_2, \dots, A_m\}$ . Define  $f_i : \mathcal{Q}_n \rightarrow \mathbb{R}$  by

$$f_i(x) = \prod_{h=1}^s \left( \sum_{j \in A_i} x_j - r_h \right) \quad \text{for } x \in \mathcal{Q}_n.$$

Note that  $f_i(A_j) = \prod_{h=1}^s (|A_i \cap A_j| - r_h) \neq 0$  if  $i = j$ , and  $f_i(A_j) = 0$  if  $i \neq j$ , so the functions  $f_1, f_2, \dots, f_m$  are linearly independent. Show that more is true: the set  $F$  consisting of these  $f_i$  and the functions  $x_I (\sum_{j=1}^n x_j - k)$  with  $|I| \leq s-1$  is also an independent set.

**148.** Let  $S = \{r_1, r_2, \dots, r_s\}$  and  $\mathcal{F} = \{A_1, A_2, \dots, A_m\}$ , where  $|A_1| \leq |A_2| \leq \dots \leq |A_m|$ . For  $1 \leq i \leq m$ , let  $a_i = \mathbf{1}_A \in \mathbb{R}^n$  be the characteristic vector of  $A_i$ ; thus  $a_i = (a_{i1}, a_{i2}, \dots, a_{in})$ , where  $a_{ij}$  is 1 if  $j \in A_i$  and 0 otherwise. Also, for  $x, y \in \mathbb{R}^n$  write  $\langle x, y \rangle$  for the inner product of the vectors  $x$  and  $y$ , so that  $\langle a_i, a_j \rangle = |A_i \cap A_j|$ .

For  $1 \leq i \leq m$ , set

$$f_i(x) = f_i(x_1, \dots, x_n) = \prod_{j: r_j < |A_i|} (\langle a_i, x \rangle - r_j),$$

so that  $f_i(x)$  is a polynomial in  $\mathbb{R}[x_1, \dots, x_n]$ . Observe that

$$f_i(a_j) = 0 \quad \text{for } 1 \leq j < i \leq m,$$

and

$$f_i(a_i) \neq 0 \quad \text{for } 1 \leq i \leq m,$$

so the polynomials  $f_1, f_2, \dots, f_m$  are independent over  $\mathbb{R}$ . Reduce the degrees of these polynomials in a suitable manner to preserve their independence and to make it obvious that the subspace of  $\mathbb{R}[x]$  they span has dimension at most

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{s}.$$

**149.** Apply the result in Problem 138 concerning hobnails.

**150.** Follow the proof of the Lusternik–Schnirelmann–Borsuk theorem from Problem 41.

**154.** Let  $G$  be the *friendship graph*: the vertices are the people in the party, and an edge means acquaintance. Then the *friendship condition* is that any two vertices have precisely one common neighbour. Equivalently, every edge is in a triangle, the diameter is at most two, and there is no quadrilateral.

Suppose that no vertex is joined to every other vertex. Show first that  $G$  is  $k$ -regular for some  $k \geq 3$  and  $n = k(k-1) + 1$ .

Turning to the heart of the problem, consider the number of closed walks of length  $p$  for some prime  $p$  dividing  $k-1$ . Prove that this number is a multiple of  $p$  and also that it is congruent to 1 modulo  $p$ .

**155.** Suppose that we have a fixed-point-free polarity. Describe it in terms of graphs.

**156.** Prove the following stronger assertion. Let  $x_1, \dots, x_n$  be vectors of norm at most 1 in a  $d$ -dimensional normed space. Then there is a permutation  $\pi \in S_n$  such that

$$\left\| \sum_{i=1}^k x_{\pi(i)} - \frac{k-d}{n} x \right\| \leq d$$

for every  $k$ , where  $x = \sum_{i=1}^n x_i$ .

To prove this inequality, first find  $\pi(n)$ , then  $\pi(n-1)$ , and so on.

**158.** Call the Devil *Simple* if she never eats a square that the Angel could have jumped to earlier. More precisely, when moving after the  $t$ th move of the Angel, the Simple Devil does not eat a square within distance  $c$  of any of the first  $t-1$  positions of the Angel. Show that if the Angel can defeat the Simple Devil then she can defeat the Devil as well.

**159.** Line has a winning strategy: it can force Point to stay outside larger and larger compact sets, thereby forcing convergence.



# 3

## The Solutions

### 1. The Lion and The Christian

*A lion and a Christian in a closed circular Roman arena have equal maximum speeds. Can the lion catch the Christian in finite time?*

*Solution.* At the first sight the *Hint* gives an elegant and very simple solution. Indeed, writing  $O$  for the centre of the arena,  $L$  for the lion, and  $M$  ('man') for the Christian, if  $L$  keeps on  $OM$  and approaches  $M$  at maximal speed then we may simplify the calculations by making  $M$  run along the boundary circle of radius 1. Then if  $L$  starts at the centre (which may clearly be assumed) then  $L$  will run along a circle of radius  $1/2$ , so  $L$  will catch  $M$  in the time it takes to cover distance  $\pi$ . This assertion is instantly justified by Figure 22 which shows that if the arc length  $MM'$  on the outer circle of radius 1 is  $\varphi$  then  $OPL'$  is also  $\varphi$  and hence  $OSL'$  is  $2\varphi$ , where  $S$  is the centre of the inner circle (of radius  $1/2$ ) touching the outer circle in  $P$  and the line  $OM$  in  $O$ . Consequently, in the time it takes the man to get from  $M$  to  $M'$  on the boundary circle, the lion gets from  $L = O$  to  $L'$ . Hence  $L$  catches  $M$  in  $P$ . (Equivalently, the angles marked  $\psi$  and  $2\psi$  show that the arc length  $M'P$  on the boundary circle is precisely the arc length  $L'P$  on the inner circle.)

At this point, the reader should stop to decide whether this is indeed right or not.

In fact, the argument above is incorrect: not only did we fail to justify the assumption that  $M$  should run along the boundary circle, but this assumption is totally unjustifiable. As shown by A.S. Besicovitch in 1952, and presented by Littlewood in his *Miscellany*, the Christian can escape if he keeps away from the boundary and runs along a suitable polygonal path.

We may clearly assume that when the chase starts, the man is at an interior point  $M_0$  of the circular arena of radius 1; to keep track of the distances, write  $1 - \varepsilon < 1$  for the distance  $OM_0$ . Suppose that we have constructed  $M_{k-1}$ , and when the man reaches  $M_{k-1}$ , the lion is at  $L_{k-1} \neq M_{k-1}$ . Let  $M_k$  be such that  $M_{k-1}M_k$  is perpendicular to  $OM_{k-1}$ , has length  $\ell_k$ , say, and  $L_{k-1}$  is *not* in the interior of the

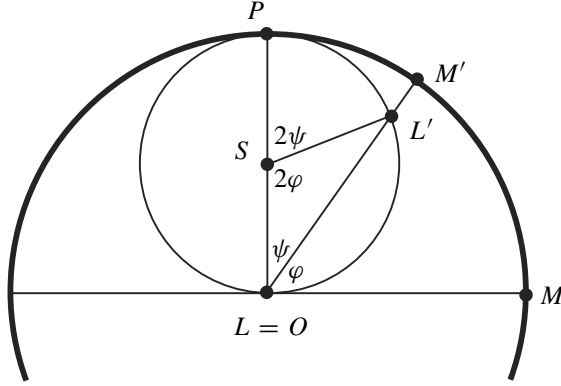


Fig. 22. The positions of the lion and man, with the man running along the circumference and the lion keeping to the radius  $OM$ .

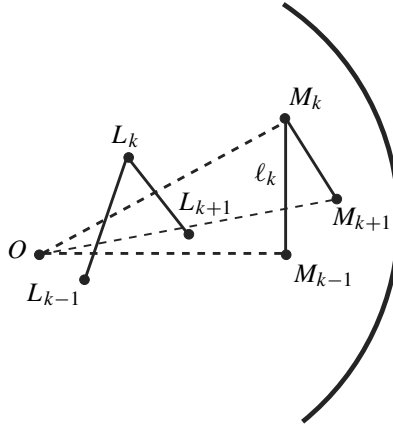


Fig. 23. The polygonal path of the man.

open half-plane determined by the line  $OM_{k-1}$  containing  $L_{k-1}$ , as in Figure 23. While the Christian runs along the segment  $M_{k-1}M_k$  from  $M_{k-1}$  to  $M_k$  (at full speed, of course), he cannot be eaten, and when he gets to  $M_k$ , the lion is at some point  $L_k \neq M_k$ . The total length of the polygonal path  $M_0M_1M_2 \dots$  is  $\sum_{k=1}^{\infty} \ell_k$ ; if this sum is infinite then the Christian will not be eaten while running along this path, so he will never be eaten: he will run forever. But can we ensure that there is such a polygonal path inside the arena? The square of the distance  $OM_n$  is clearly  $(1 - \varepsilon)^2 + \sum_{k=1}^n \ell_k^2 \leq 1 - \varepsilon + \sum_{k=1}^n \ell_k^2$ , so the path  $M_0M_1M_2 \dots$  will stay inside the arena if  $\sum_{k=1}^{\infty} \ell_k^2 < \varepsilon$ .

All that remains is to choose a suitable sequence  $\ell_1, \ell_2, \dots$ . Set  $\ell_k = ck^{-3/4}$  for  $c = \varepsilon/2$ . Then, trivially,  $\sum_{k=1}^{\infty} \ell_k$  is infinite, and  $\sum_{k=1}^{\infty} \ell_k^2 = c^2 \sum_{k=1}^{\infty} k^{-3/2} < c^2(1 + \int_1^{\infty} x^{-3/2} dx) = 3c^2 < \varepsilon$ , so the sequence  $(\ell_k)$  does have the required properties. [The actual choice of  $c$  is utterly unimportant; it is clear that  $\ell_k = ck^{-3/4}$  will do if  $c$  is a small enough positive constant. Another natural choice is  $\ell_k = (k + s)^{-3/4}$  for  $s$  large enough.]  $\square$

**Notes.** This problem was invented by Richard Rado in the late 1930s, and was popularized by Littlewood, who devoted to it four pages of his *Miscellany*. Rado, who was an exceptionally tactful man, called the Christian in his problem a ‘man’, so the problem ran under the title ‘Lion and Man’.

Years after Besicovitch had found the solution above, Croft investigated several variants of the problem. For example, he proved that if the man has to run along a curve of uniformly bounded curvature then he will be eaten by a clever lion, but not by one who stays on the radius  $OM$ . Also,  $n$  lions can catch a man in an  $n$ -dimensional ball, but  $n - 1$  cannot.

I believe that the following problem mentioned by Littlewood is still unsolved. Can two lions catch a man in a bounded area with rectifiable lakes?

B. Bollobás (ed.), *Littlewood's Miscellany*, Cambridge University Press (1986), 200 pp.  
 H.T. Croft, ‘Lion and man’: a postscript, *J. London Math. Soc.* **39** (1964), 385–390.

## 2. Integer Sequences: Erdős Problems for Epsilons

- (i) Among any  $n + 1$  positive integers none of which is greater than  $2n$  there are two such that one divides the other.
- (ii) Among any  $n + 1$  positive integers none of which is greater than  $2n$  there are two that are relatively prime.
- (iii) Suppose that we have  $n$  natural numbers none of which is greater than  $2n$  such that the least common multiple of any two is greater than  $2n$ . Then all  $n$  numbers are greater than  $2n/3$ .
- (iv) For  $r, s \geq 1$ , every sequence of  $n = rs + 1$  distinct integers has an increasing subsequence of length  $r + 1$  or a decreasing subsequence of length  $s + 1$ .

*Proof.* (i) Let  $1 < a_1 < a_2 < \cdots < a_{n+1} \leq 2n$  be integers. Write each  $a_i$  as  $2^{k_i} b_i$ , where  $k_i$  and  $b_i$  are integers, with  $k_i$  non-negative and  $b_i$  odd. Then each  $b_i$  is one of  $n$  odd numbers, namely  $1, 3, \dots, 2n - 1$ , so some two of them are equal:  $b_i = b_j$ , say, where  $i \neq j$ . Assuming, as we may, that  $k_i < k_j$ , we find that  $a_i$  divides  $a_j$ .

(ii) Among any  $n + 1$  integers from  $1$  to  $2n$ , there are two consecutive numbers (since, for example, one of the  $n$  pairs  $\{1, 2\}, \{3, 4\}, \dots, \{2n - 1, 2n\}$  contains two of the  $n + 1$  numbers); any two consecutive numbers are relatively prime.

(iii) Let  $a_1 < a_2 < \cdots < a_k \leq 2n$  be integers such that the least common multiple of any two is greater than  $2n$ . Suppose that  $a_1 \leq 2n/3$ . Then  $2a_1, 3a_1, a_2, a_3, \dots, a_k$  are  $k + 1$  numbers between  $2$  and  $2n$ , none dividing another. The assertion in (i) implies that  $k + 1 \leq n$ .

(iv) Apply induction on  $r$ , say. For  $r = 1$  the assertion is obvious, so let us turn to the induction step. Suppose that  $r > 1$  and the assertion is true for smaller values of  $r$ . Let  $(a_i)_{i \in [n]}$  be a sequence of  $n$  distinct integers, and let  $L$  be the set of suffices of the last terms of increasing subsequences of length  $r$ :

$$L = \{i_r : \text{there are } i_1 < \cdots < i_r \text{ with } a_{i_1} < \cdots < a_{i_r}\}.$$

If  $|L| \geq s + 1$  then either  $(a_\ell)_{\ell \in L}$  is a decreasing sequence of length  $|L| \geq s + 1$ , or  $a_\ell < a_{\ell'}$  for some  $\ell, \ell' \in L$  with  $\ell < \ell'$ . In the latter case an increasing sequence of length  $r$  ending in  $a_\ell$  may be continued with  $a_{\ell'}$  to give us an increasing sequence of length  $r + 1$ .

On the other hand, if  $|L| \leq s$  then the subsequence  $(a_i)_{i \in [n] \setminus L}$  has at least  $rs + 1 - s = (r - 1)s + 1$  terms, and has no increasing subsequence of length  $r$ . Consequently, by the induction hypothesis,  $(a_i)_{i \in [n] \setminus L}$  has a decreasing subsequence of length  $s + 1$ . □





Fig. 24. Paul Erdős, his mother, and the thirteen-year-old Lajos Pósa.

**Notes.** Paul Erdős made a point of meeting budding mathematicians, *epsilons*, and ask them mathematical questions needing only ingenuity to solve; the five assertions above were among his favourite questions. In particular, Erdős asked the first one of the thirteen-year-old Lajos Pósa during one of their first meetings, while eating soup, and Pósa came up with this solution immediately, prompting the remark that on this occasion champagne would have been more appropriate than soup.

### 3. Points on a Circle

(i) Let  $X$  and  $Y$  be subsets of the vertex set of a regular  $n$ -gon. Show that there is a rotation  $\varrho$  of this  $n$ -gon such that  $|X \cap \varrho(Y)| \geq |X||Y|/n$ .

(ii) Let  $S = F_1 \cup F_2$ , where  $F_1$  and  $F_2$  are closed subsets of  $S$ , the unit circle in  $\mathbb{R}^2$ . Then for some  $j$ ,  $1 \leq j \leq 2$ , and every angle  $0 \leq \alpha \leq \pi$ , the set  $F_j$  contains two points  $x$  and  $y$ , with  $y$  obtained from  $x$  by rotating it through  $\alpha$ .

(iii) Every set of  $n \geq 1$  non-zero integers has a sum-free subset with more than  $n/3$  elements.

*Proof.* (i) For a rotation  $\varrho$ , set  $f(\varrho) = |X \cap \varrho(Y)|$ , so that our task is to prove that  $\max_{\varrho} f(\varrho) \geq |X||Y|/n$ . Clearly, for every pair  $(x, y) \in X \times Y$ , there is a (unique) rotation  $\varrho$  with  $x = \varrho(y)$ . Hence,  $\sum_{\varrho} f(\varrho) = |X||Y|$ , where the summation is over all  $n$  rotations. Consequently,  $\max_{\varrho} f(\varrho) \geq \frac{1}{n} \sum_{\varrho} f(\varrho) = |X||Y|/n$ .

(ii) Divide the circle  $S$  into  $n = 2^m$  arcs of the same length; let these arcs be  $I_1, \dots, I_n$  in a cyclic order. For  $j = 1, 2$ , let

$$X_j^{(m)} = \{k \in [n] : (I_k \cup I_{k+1}) \cap F_j \neq \emptyset\},$$

where  $[n] = \{1, \dots, n\}$  and the indices  $k$  are taken modulo  $n$ . Then  $|X_1^{(m)}| + |X_2^{(m)}| > n$ , so  $|X_j^{(m)}| > n/2 = 2^{m-1}$  for some  $j = j(n)$ . In fact, if for  $m > 2$  we have  $|X_j^{(m)}| > 2^{m-1}$  then  $|X_j^{(m-1)}| > 2^{m-2}$  holds as well, so the same  $j$  will do for every  $j(m)$ . Indeed, the  $j$  that occurs infinitely often as a  $j(m)$  is like that.

We claim that the set  $F_j$  has the required properties. To see this, let  $\alpha = \rho\pi$ , with  $0 \leq \rho \leq \pi$ . Given  $n = 2^m$ , set  $r = \lfloor \rho 2^{m-1} \rfloor$ . By part (ii), our set  $X_j^{(m)}$  contains two points at distance  $r$ : for some  $k \in [n]$ , we have  $k, k+r \in X_j^{(m)}$ . Consequently, there are points  $x_m, y_m \in F_j$  such that the angle  $\beta_m$  they determine is close to  $\alpha$ :  $|\beta_m - \alpha| < \pi/2^{m-2}$ . Since the circle  $S$  is compact, for some sequence  $(m_i)$  and points  $x, y \in S$  we have  $x_{m_i} \rightarrow x$  and  $y_{m_i} \rightarrow y$ . Clearly, the angle determined by  $x$  and  $y$  is precisely  $\alpha$ . Finally, as the set  $F_j$  is closed, it contains  $x$  and  $y$ .  $\square$

(iii) Let  $p = 3m + 2$  be a prime not dividing any element of  $A$ . For  $k = 1, \dots, p-1$ , let  $A_k$  be the set of elements  $a$  of  $A$  such that, modulo  $p$ ,  $ka$  is congruent to one of  $m+1, m+2, \dots, 2m+1$ . As the set  $\{m+1, m+2, \dots, 2m+1\}$  is sum-free modulo  $p$ , so is  $A_k$ . Note that, for every  $a \in A$ , the sequence  $a, 2a, \dots, (p-1)a$ , taken modulo  $p$ , is a permutations of  $1, 2, \dots, p$ ; consequently,  $\sum_{k=1}^{p-1} |A - k| = n(m-1)$ . Hence,  $|A_\ell| \geq n(m+1)/(p-1) =$

$n(m+1)/(3m+1)$  for some  $\ell$ . This set  $A_\ell$  is a sum-free subset of  $A$  with at least  $\lceil (n+1)/3 \rceil$  elements.

**Notes.** Parts (i) and (ii) are simple examples of the use of a basic tool of combinatorics (and analysis), *double counting* (interchanging the order of integration). In fact, the proof of (i) is just about the first step in probabilistic combinatorics: let  $\varrho$  be a random rotation (with the uniform distribution) and take the expectation of  $|X \cap \varrho(Y)|$ :

$$\max_{\varrho} |X \cap \varrho(Y)| \geq \mathbb{E}(|X \cap \varrho(Y)|) = |X||Y|/n.$$

It is trivial to extend the assertions to measurable sets with the normalized Lebesgue measure on  $S = \mathbb{R}/\mathbb{Z}$ , and, more generally, to the Haar measure on a compact metric group, or a metric probability space with a transitive group of measure-preserving isometries. Here are two obvious ‘extensions’ to the circle group  $S = \mathbb{R}/\mathbb{Z}$ . Identifying  $S$  with  $[0, 1)$ , rotation through  $\varrho \in S$  corresponds to addition modulo 1: for  $\varrho \in S = [0, 1)$  we take  $\varrho(x) = x + \varrho$  modulo 1.

1. Let  $X$  and  $Y$  be measurable subsets of  $S$ . Then for some  $\varrho \in S$ , i.e.,  $0 \leq \varrho < 1$ ,

$$\mu(X \cap (Y + \varrho)) = \mu(X \cap \varrho(Y)) \geq \mu(X)\mu(Y).$$

2. Let  $X$  and  $Y$  be subsets of  $S$ , with  $X$  measurable and  $Y$  finite. Then

$$|X \cap \varrho(Y)| \geq \lceil \mu(X)|Y| \rceil$$

for some rotation  $\varrho$ . If, in addition,  $X$  is closed then for some  $\varrho$  we have

$$|X \cap \varrho(Y)| \geq \lfloor \mu(X)|Y| \rfloor + 1.$$

To see the last inequality, note that, as  $Y$  is closed,

$$\max_{\varrho} |X \cap \varrho(Y)| = \min_{\varepsilon > 0} \max_{\varrho} \lceil |X_\varepsilon \cap \varrho(Y)| \rceil,$$

where  $X_\varepsilon = \{z \in S : d(z, x) < \varepsilon \text{ for some } x \in X\}$ , with  $d(z, x)$  the distance between  $z$  and  $x$ , as points of  $\mathbb{R}/\mathbb{Z}$ .

Part (iii) is another Erdős problem for epsilons.

## 4. Partitions into Closed Sets

*For  $n \geq 1$ , the space  $\mathbb{R}^n$  cannot be partitioned into countably many (but at least two) closed sets.*

*Proof.* If for some  $n \geq 2$  the set  $\mathbb{R}^n$  can be partitioned into countably many closed sets then so can  $\mathbb{R}$ . To see this, just pick a line that meets at least two of the closed sets: then the intersections of the closed sets partition our line into closed sets.

Let us show then that  $\mathbb{R}$  cannot be partitioned into countably many (but at least two) closed subsets of  $\mathbb{R}$ . Suppose for a contradiction that  $\mathbb{R} = \bigcup_{k=1}^{\infty} F_k$ , with each  $F_k$  closed,  $F_k \cap F_\ell = \emptyset$  for all  $1 \leq k < \ell < \infty$ , and  $F_1$  and  $F_2$  non-empty.

We may assume that there is a closed interval  $I_1 = [a_1, b_1]$  such that  $a_1 \in F_1$ ,  $b_1 \in F_2$  and  $(a_1, b_1) \cap (F_1 \cup F_2) = \emptyset$ . We shall construct two sequences of reals,  $a_1 < a_2 < \dots < b_2 < b_1$ , and two sequences of natural numbers,  $m_1 = 1 < n_1 = 2 < m_2 < n_2 < \dots$  such that  $a_k \in F_{m_k}$ ,  $b_k \in F_{n_k}$  and  $(a_k, b_k) \cap E_k = \emptyset$ , where  $(a_k, b_k)$  is the open interval from  $a_k$  to  $b_k$ , and  $E_k = \bigcup_{i=1}^{n_k} F_i$ . This will prove the result since then the intersection  $\bigcap_{k=2}^{\infty} I_k$  of the closed intervals  $I_k = [a_k, b_k] \subset (a_{k-1}, b_{k-1})$  is non-empty and disjoint from every  $F_i$ .

To construct these sequences, suppose that we have chosen  $a_1 < a_2 < \dots < a_k < b_k < \dots < b_2 < b_1$  that satisfy our conditions. Since  $(a_k, b_k) \cap E_k = \emptyset$ , there is a minimal index  $m_{k+1}$  greater than  $n_k$  such that  $G_{k+1} = F_{m_{k+1}} \cap (a_k, b_k) \neq \emptyset$ . Let  $a_{k+1}$  be the maximum of  $G_{k+1}$ ; then  $a_k < a_{k+1} < b_k$ . Next, let  $n_{k+1}$  be the minimal index such that  $H_{k+1} = F_{n_{k+1}} \cap (a_{k+1}, b_{k+1}) \neq \emptyset$ , and let  $b_{k+1}$  be the minimum of  $H_{k+1}$ . The sequences constructed in this way have the required properties.  $\square$

## 5. Triangles and Squares

*The maximal area of a triangle contained in a unit square is  $1/2$ , and the minimal area of a triangle containing a unit square is 2.*

*Proof.* It is easily seen (by appealing to compactness or from first principles) that there is indeed a triangle of maximal area inscribed in a unit square, and there is also a circumscribed triangle of minimal area.

Let  $ABCD$  be a unit square, with  $PQR$  an inscribed triangle of maximal area. First of all, the vertices  $P$ ,  $Q$  and  $R$  are on the sides of the square. Second, suppose that  $P$  is an interior point of the side  $AB$ . When moving  $P$  to either  $A$  or  $B$  the area of  $PQR$  does not decrease. Consequently, we may assume that  $P$ , and so each vertex of the triangle, is also a vertex of the square. But then the area of the triangle is exactly  $1/2$ , as claimed.

It is only a little more cumbersome to prove that the minimum of the area of a circumscribed triangle is 2, as in Figure 25.

To see this, first note that to minimize the area of a triangle  $PQR$ , with  $P$  and the half-lines  $PQ$  and  $PR$  fixed, in which the side  $QR$  is demanded to pass through a fixed point  $A$  in the interior of the angle  $QPR$ , as in Figure 26, we have to choose  $QR$  in a such a way that  $A$  becomes its midpoint. Indeed, if  $A$  is not the midpoint of  $QR$ , say,  $QA > AR$ , then by moving  $Q$  slightly closer to  $P$  (and so  $R$  away from  $P$ ), the area of the triangle  $PQR$  decreases.

From this it follows that if  $PQR$  is a triangle of minimal area containing a unit square then the three midpoints of  $PQR$  belong to the square. Consequently, a side of the unit square is on a side of the triangle, and the opposite side of the square joins two midpoints of the triangle, as in Figure 25. In particular, the triangle has area 2, as claimed.  $\square$

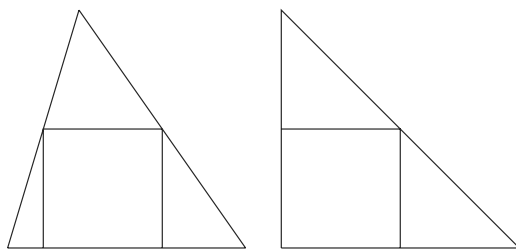


Fig. 25. Triangles of minimal area circumscribed about a unit square.

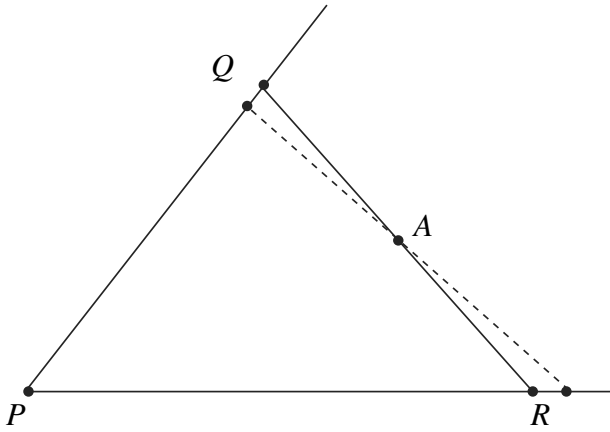


Fig. 26. Maximizing the area of a triangle cut off from an angle.

**Notes.** The arguments above give us all the extremal arrangements as well. Also, it is immediate to extend these arguments to solve the analogous problems for a regular  $n$ -gon of unit area and a triangle.

## 6. Polygons and Rectangles

*Every convex polygon of area 1 is contained in a rectangle of area 2.*

*Proof.* Let  $K$  be a convex polygon of area 1. This polygon contains two points  $a$  and  $b$  at maximal distance:  $d(a, b) = \text{diam}(K)$ . Let  $\ell_a$  be a line through  $a$  perpendicular to  $ab$ , and define  $\ell_b$  analogously. Then the strip determined by the lines  $\ell_a$  and  $\ell_b$  contains  $K$ . Let  $m_c$  and  $m_d$  be the lines perpendicular to the line  $\ell_a$  (and so to  $\ell_b$ ) that touch  $K$  on the two sides, say, at points  $c$  and  $d$ , so that the strip determined by these lines also contain  $K$  (see Figure 27). Then the rectangle  $R$  determined by the lines  $\ell_a$ ,  $\ell_b$ ,  $m_c$  and  $m_d$  contains  $K$ . The union of the triangles  $abc$  and  $abd$  is contained in  $K$ , and the area of this union is precisely half of the area of  $R$ .  $\square$

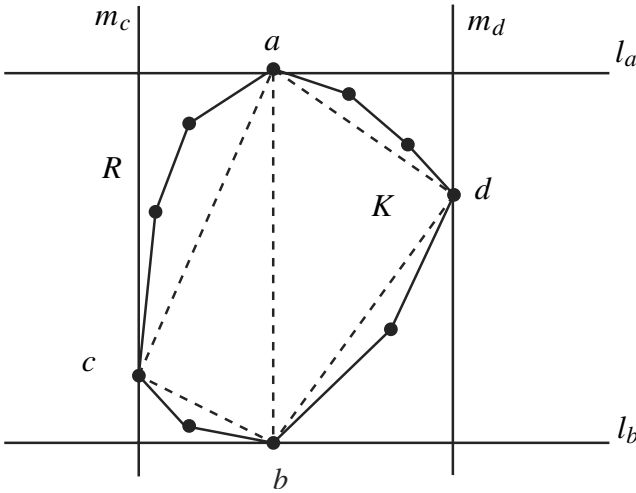


Fig. 27. A rectangle  $R$  containing a polygon  $K$ .

## 7. African Rally

A car circuit is going through the towns  $T_1, T_2, \dots, T_n$  in this cyclic order (so from  $T_i$  the route leads to  $T_{i+1}$  and from  $T_n$  back to  $T_1$ ), and a car is to travel around this circuit, starting from one of the towns. At the start of the journey, the tank of the car is empty, but in each town  $T_i$  it can pick up  $p_i$  amount of fuel.

(i) If  $\sum_{i=1}^n p_i$  is precisely sufficient to drive round the entire circuit then there is a town such that if the car starts from there then it can complete the entire circuit without running out of fuel.

(ii) If each section of the circuit (from town  $T_i$  to  $T_{i+1}$ ) needs an integer amount of fuel, each  $p_i$  is an integer and  $\sum_{i=1}^n p_i$  is precisely one more than the amount of fuel needed to drive round the entire course, then there is precisely one town such that, starting from there, there is at least 1 unit of fuel in the tank not only at the end but throughout the circuit.

*Proof.* Let  $r_i$  be the amount of fuel required to drive from a town  $T_i$  to the next town, and set  $a_i = p_i - r_i$ . Furthermore, let us introduce the notation  $[k, \ell) = \{k, k+1, \dots, \ell-1\}$ , with the numbers taken modulo  $n$ , and no number occurring more than once. In particular,  $[k, k) = \mathbb{Z}_n$  for every  $k$ . Also, set  $S[k, \ell) = \sum_{i \in [k, \ell)} a_i$ , so that  $S[k, k) = \sum_{i=1}^n a_i$  for every  $k$ .

(i) We are told that  $S[k, k) = \sum_{i=1}^n a_i = 0$ , and what we need is that for some  $k$  all partial sums  $S[k, \ell)$ ,  $\ell = k+1, k+2, \dots, k$ , are non-negative. Suppose for a contradiction that this is not the case, i.e., for every  $k$  there is an  $\ell \neq k$  such that  $S[k, \ell) < 0$ . Let then  $k_0 = 1 < k_1 < k_2 < \dots$  be such that  $S[k_i, k_{i+1}) < 0$  for every  $i$ . By the pigeonhole principle, there are indices  $0 \leq i < j \leq n$ , such that  $k_i = k_j$ . But then in the sum  $S[k_i, k_{i+1}) + S[k_{i+1}, k_{i+2}) + \dots + S[k_{j-1}, k_j)$  every  $a_s$  occurs the same number, say,  $m$ , times, giving us the contradiction that

$$0 > S[k_i, k_{i+1}) + S[k_{i+1}, k_{i+2}) + \dots + S[k_{j-1}, k_j) = m \sum_{s=1}^n a_s = 0.$$

(ii) The exact analogue of the argument in (i) shows that for some  $k$  we have  $S[k, \ell) \geq 1$  for every  $\ell \neq k$ . Uniqueness is even quicker: since  $S[k, \ell) + S[\ell, k) = \sum_{i=1}^n a_i = 1$ , for any two values  $k \neq \ell$  one of the sums  $S[k, \ell)$ ,  $S[\ell, k)$  is at least 1 and the other is at most 0. In particular, there is at most one  $k$  such that all sums  $S[k, \ell)$ ,  $k \neq \ell$  are at least 1.  $\square$

**Notes.** This was one of my favourite problems I used to ask from students when I interviewed them for admission to Trinity College, Cambridge.

The pigeonhole principle is perhaps the most basic tool of combinatorics; see, e.g., p. 4 of G.E. Martin, *Counting: the Art of Enumerative Combinatorics*



(Springer, 2001); pp. 25–27 of M.J. Erickson, *Introduction to Combinatorics* (Wiley, 1996), or P.J. Cameron, *Combinatorics: Topics, Techniques, Algorithms* (Cambridge University Press, 1994). In fact, the pigeonhole principle was used tacitly in the proofs of parts (i) and (ii) of Problem 2 as well.



## 8. Fixing Convex Domains

*Given a convex open set  $D$  in the plane, if a set  $N$  (of ‘nails’) fixes  $D$  but no proper subset of it does, then  $N$  consists of at most six points. Here six is the minimum: there are sets  $N$  with six points.*

*Proof.* First, let us clarify the assertion above. A set  $N$  is said to *fix*  $D$  if it is disjoint from  $D$ , and for every direction (angle)  $\varphi$  there is an  $\varepsilon > 0$  such that if  $D$  is translated in the direction  $\varphi$  through a distance which is strictly positive but less than  $\varepsilon$  then some point of  $N$  will be in the translated set. Call a fixing set *primitive* if no proper subset of it fixes our body.

The assertion is that the *maximal cardinality of a primitive fixing set* of a domain is six. Here the maximum is taken over all domains and all primitive fixing sets.

The six vertices of a regular hexagon (or any hexagon whose opposite sides are parallel) form a primitive fixing set of the open hexagon, so we see that the maximum is at least six (see Figure 28).

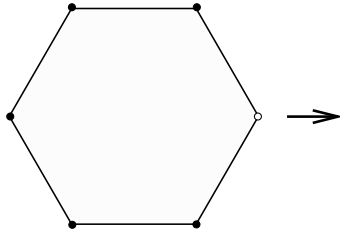


Fig. 28. The vertex set of a regular hexagon is a primitive fixing set.

It is also easy to show that a primitive fixing set  $N$  cannot contain more than six points. Indeed, suppose that  $N = \{x_1, \dots, x_n\}$ , with  $n \geq 7$ . For every point  $x_i$  of there is an open circular arc  $I_i = (a_i, b_i)$  of the unit circle such that  $x_i$  (by itself) prevents translation of  $D$  in a direction  $\varphi$  if and only if  $\varphi$  belongs to  $I_i$ . Now,  $N$  fixes  $D$  if the sets  $I_1, I_2, \dots, I_n$  cover the entire circle unit circle  $C$ .

If  $N$  is primitive, then no  $I_i$  is covered by the union of the other arcs. Let us observe that this implies that no point of the circle is contained in any three of the open arcs  $I_1, I_2, \dots, I_n$ . To see this, suppose that  $I_i, I_j$  and  $I_k$  have a point in common. Then the union of these intervals is again an interval. This union has to be a proper open arc  $(a, b)$  since if it were the entire circle then we would have to have  $n \leq 3$ . Now, if the end-points of  $I_i$  and  $I_j$ , say, contain the points  $a$  and  $b$  then  $I_k$  is contained in  $I_i \cup I_j$ . This contradiction shows that no three of the arcs have a point in common.

All that remains to check is that for  $n \geq 7$  some point of the circle is in at least three of the arcs. The points  $x_1, x_2, \dots, x_n$  are on the boundary of  $D$ , so form a convex  $n$ -gon  $P$  (see Figure 29).

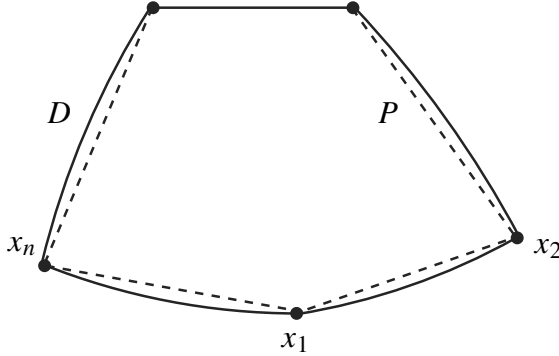


Fig. 29. A domain  $D$  with nails  $x_1, \dots, x_n$ , and the corresponding polygon  $P$ . The domain is smooth at  $x_1$ , but not at the other vertices.

Each arc  $I_i$  is at least as large as the angle of  $P$  at  $x_i$ . Hence, the total length of the arcs  $I_1, I_2, \dots, I_n$  is at least  $(n - 2)\pi$ . Therefore, if  $n \geq 7$ , the total length is more than  $4\pi$ ; consequently, some point of the circle is contained in three of the open arcs, completing the proof.  $\square$

**Notes.** The simple result in this problem was noticed by Tomor. The proof above shows that if  $N$  is a minimal set of six points fixing a convex domain  $D$  then  $D$  is a hexagon whose opposite sides are parallel, and  $N$  is the set of vertices of this hexagon. It is also easy to see that every convex planar domain can be fixed by three points. Furthermore, as the argument above shows, if we use only points at which the boundary of our convex set is smooth, then a primitive fixing set has at most four points.

In fact, primitive fixing systems have been studied for almost hundred years. Let  $f(K)$  be the maximal cardinality of a primitive fixing system for a convex body  $K$ , and let  $f(d)$  be the maximum of  $f(K)$  for  $d$ -dimensional convex bodies. Steinitz proved in 1914 that  $f(K) \leq 2d$  if  $K \subset \mathbb{R}^d$  is smooth, and almost fifty years later Fejes Tóth continued the study of primitive fixing systems of smooth three-dimensional bodies. Tomor noted that the rhombic dodecahedron (see Figure 30) shows that  $f(3) \geq 14$  and, in view of this example, speculated that  $f(3) = 14$ . In his review of Fejes Tóth's paper, L.W. Danzer remarked that, writing  $Q^d = \{x = (x_i) \in \mathbb{R}^d : 0 \leq x_i \leq 1 \text{ for every } i\}$  for the unit cube, the convex hull  $K$  of  $Q^d \cup (-Q^d)$  shows that  $f(d) \geq d(2^d - 1)$ . This example led Danzer to the conjecture that  $f(d)$  is, in fact,  $d(2^d - 1)$ .

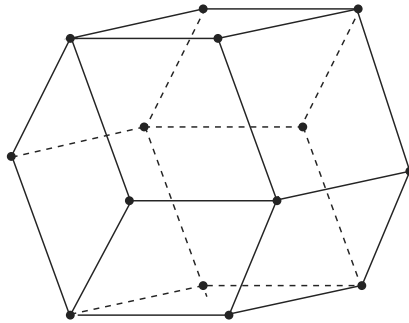


Fig. 30. The fourteen vertices of a rhombic dodecahedron form a primitive fixing set, so  $f(3) \geq 14$ .

As it happens, for  $d \geq 3$  the value  $f(d)$  is not even finite (apologies for the misleading definition): Bollobás showed that there are convex 3-dimensional bodies that have primitive fixing systems of continuum cardinality. Further (and often rather technical) related results have been proved by Boltyanski, Morales Amaya and Martini.

B. Bollobás, Fixing system for convex bodies, *Studia Sci. Math. Hungar* **2** (1967), 351–354.  
 V.G. Boltyanski and E. Morales Amaya, Minimal fixing systems for convex bodies, *J. Appl. Anal.* **1** (1995), 1–11.

V. Boltyanski and E. Morales Amaya, Cardinalities of primitive fixing systems for convex bodies, *The Branko Grünbaum birthday issue, Discrete Comput. Geom.* **24** (2000), 209–218.

V. Boltyanski and H. Martini, Carathéodory's theorem and  $H$ -convexity, *J. Combin. Theory Ser. A* **93** (2001), 292–309.

V. Boltyanski, Solution of the illumination problem for bodies with  $\text{md}M = 2$ , *Discrete Comput. Geom.* **26** (2001), 527–541.

L. Fejes Tóth, On primitive polyhedra, *Acta Math. Acad. Sci. Hungar.* **13** (1962), 379–382.

E. Steinitz, *J. Reine Angew. Math.* **144** (1914), 1–40.

B. Tomor, A problem of fixing convex domains (in Hungarian), *Matematikai Lapok* **13** (1963), 120–123.

## 9. Nested Subsets

*Every infinite set contains uncountably many nested subsets.*

*Proof.* Let  $\mathcal{F} = \{A_\gamma : \gamma \in \mathbb{R}\}$ , where  $A_\gamma = \{q \in \mathbb{Q} : q < \gamma\}$ , so that  $\mathcal{F}$  is a continuum family of subsets of the countable set  $\mathbb{Q}$ , the set of rationals. Then for  $\gamma < \delta$  we have  $A_\gamma \neq A_\delta$  and  $A_\gamma \subset A_\delta$ .  $\square$

**Notes.** This was one of my standard questions for freshmen in Cambridge. I was always sorry that the above elegant (essentially one-line) solution could be given, since it gives the impression that one has to hit upon a ‘trick’, rather than work a solution out slowly. So whenever my supervisees failed to solve this problem, first I gave a pedestrian solution along the following lines.

Let us be a little more ambitious, and try to construct a family of nested subsets of  $\mathbb{N}$  indexed by 0-1 sequences or, as we shall do it, by the infinite branches of the rooted infinite binary tree, the rooted tree in which each vertex has two children. (In fact, as we do not know how to construct an uncountable set whose cardinality is smaller than that of the continuum, if we are bent to give a construction, we are not ambitious at all.) Having seen the one-line solution, it is easy to say that all we do is imitate that solution, but a student who fails to solve the problem and hears this workmanlike solution, does tend to appreciate it.

In the construction below, all we shall use is that every infinite set has a partition into four infinite sets. (Needless to say, two would do just as well.) Let  $T$  be the infinite binary tree with root  $\alpha$ . First, we shall assign a pair of disjoint infinite sets,  $(U_\omega, W_\omega)$ , to each vertex  $\omega$ . We start by choosing a partition  $U_\alpha \cup W_\alpha$  of  $\mathbb{N}$  into two *infinite* sets. Having assigned two disjoint infinite sets  $U_\beta$  and  $W_\beta$  to a vertex  $\beta$  with children  $\gamma$  and  $\delta$ , partition  $U_\beta$  into four infinite sets,  $U_\beta^{(i)}$ ,  $i = 1, \dots, 4$ , and set  $W_\gamma = U_\beta^{(1)}$ ,  $U_\gamma = U_\beta^{(2)}$ ,  $W_\delta = U_\beta^{(1)} \cup U_\beta^{(2)} \cup U_\beta^{(3)}$  and  $U_\delta = U_\beta^{(4)}$ .

To complete the construction, for each infinite branch (infinite path starting in the root) take the union of the  $W$  sets assigned to the vertices of the path. By construction, if two branches part in a vertex  $\beta$  as above, with children  $\gamma$  and  $\delta$ , then the union of the  $W$  sets assigned to the branch through  $\delta$  will strictly contain the union assigned to the branch through  $\gamma$ .

Note that the only role of  $U_\beta^{(3)}$  was to make sure that the set assigned to the branch through  $\gamma$  is *strictly* contained in the set assigned to any branch through  $\delta$ , so it suffices to demand that it be non-empty.

Let us describe essentially the same pedestrian solution in a slightly more formal way. Let  $X = \{0, 1\}^{\mathbb{N}}$  be the set of all infinite 0-1 sequences, and let  $A$  be a countably infinite set. We shall interpret a sequence  $x = (x_1, x_2, \dots) \in X$  as a sequence of instructions to construct a set  $A_x \subset A$ : at each stage, 0 tells us not to

increase our current set, and 1 instructs us to increase it by an infinite set. Our aim is to make sure that once we have been instructed not to add a set, this ‘deficit’ can not be made up by later additions. To guarantee that we do not add too much, we work with two sets, a lower bound and an upper bound for the end-result.

Suppose then that  $x = x^n = (x_i)_1^n \in \{0, 1\}^n$  and we have constructed two sets,  $B_x$  and  $C_x$ , with  $B_x \subset C_x \subset A$  and  $C_x \setminus B_x$  infinite. If  $n = 0$ , so that  $x$  is the empty sequence, then  $B_x = \emptyset$  and  $C_x = A$  will do. How shall we choose  $B_y$  and  $C_y$  for the two one-term extensions of  $x$ , i.e., for  $y = x' = (x_1, \dots, x_n, 0)$  and  $y = x'' = (x_1, \dots, x_n, 1)$ ?

Pick sets  $D_x$  and  $E_x$  such that  $B_x \subset D_x \subset E_x \subset C_x$ ,  $D_x \neq E_x$ , and  $D_x \setminus B_x$  and  $C_x \setminus E_x$  are infinite. Then define  $B_{x'} = B_x$ ,  $C_{x'} = D_x$ , and  $B_{x''} = E_x$ ,  $C_{x''} = C_x$ . Finally, for  $x \in X = \{0, 1\}^{\mathbb{N}}$ , set

$$A_x = \bigcup_{n=1}^{\infty} B_{x^n},$$

where  $x^n = (x_1, \dots, x_n)$  is the length  $n$  initial segment of  $x = (x_1, \dots)$ . By this construction, if  $x, y \in X$ , with  $x_i = y_i$  for  $i \leq n$  and  $x_{n+1} = 0 < y_{n+1} = 1$ , then

$$A_x \subset D_{x^n} \text{ and } E_{x^n} \subset A_y.$$

In particular,  $A_x$  is a proper subset of  $A_y$ , so  $\{A_x : x \in X\}$  is an uncountable family of nested subsets of the countable set  $A$ .

## 10. Almost Disjoint Subsets

*Every infinite set contains uncountably many almost disjoint subsets.*

*Proof.* We take  $\mathbb{Q}$ , the countable set of rationals, as our infinite set. For every real number  $\gamma$ , let  $s_\gamma$  be a sequence of rationals converging to  $\gamma$ , and let  $S_\gamma$  be the set of rationals in this sequence. Then  $\{S_\gamma : \gamma \in \mathbb{R}\}$  is a continuum family of almost disjoint subsets of  $\mathbb{Q}$ .  $\square$

**Notes.** This is the ‘younger brother’ of Problem 9. Although occasionally even good freshmen could be foxed by it, I soon stopped asking it as much of the time it was found to be too easy.

Rather than going for convergent sequences, as in the proof above, I used to suggest that we take the countable set  $F$  of all *finite* 0-1 sequences as our infinite set. This very strong hint makes the question just about trivial and leads to a lightly disguised concrete choice of convergent sequences. Indeed, given an infinite 0-1 sequence  $x = (x_i)$ , let  $F_x$  be the set of all initial segments of  $x$ . (Thus if, using a trivial abbreviation,  $x = 11010001\dots$ , then  $F_x = \{1, 11, 110, 1101, 11010, 110100, \dots\}$ .) The set  $X = \{0, 1\}^{\mathbb{N}}$  of all 0-1 sequences is uncountable, in fact, has continuum cardinality, and if  $x, y \in X$ ,  $x \neq y$ , then the sets  $F_x$  and  $F_y$  are trivially almost disjoint.

## 11. Loaded Dice

*No matter what two loaded dice we have, it cannot happen that each of the sums 2, 3, ..., 12 comes up with the same probability.*

*Proof.* Suppose that we have a pair of dice such that each of the 11 possible sums 2, 3, ..., 12 occurs with the same probability,  $1/11$ . Let  $p_i$  be the probability that the first die shows  $i$ , and  $q_i$  that the second does. Then the probability that the sum is 2 is  $p_1q_1$ , and the probability that it is 12 is  $p_6q_6$ . Consequently,  $p_1q_1 = 1/11$  and  $p_6q_6 = 1/11$ . Also, the probability that the sum is 7 is *at least*  $p_1q_6 + p_6q_1$ , so  $p_1q_6 + p_6q_1 \leq 1/11$ . Hence,  $q_1 = 1/11 p_1$ ,  $q_6 = 1/11 p_6$  and

$$\frac{1}{11} \geq p_1q_6 + p_6q_1 = \frac{p_1}{11p_6} + \frac{p_6}{11p_1} = \frac{1}{11} \left( \frac{p_1}{p_6} + \frac{p_6}{p_1} \right) \geq \frac{2}{11},$$

where the last inequality follows from the simplest form of the AM-GM inequality. This contradiction completes the solution.  $\square$

**Notes.** In the early seventies I used to ask this in scholarship interviews in Trinity College, Cambridge. The better candidates usually got it, but it was invariably too difficult for the weaker ones.

There is a more elegant and ‘sophisticated’ solution based on generating functions; the students did not find it, but certainly enjoyed hearing about it. Let  $p_i$  and  $q_j$  be as above, and set  $P(x) = \sum_{i=1}^6 p_i x^{i-1}$  and  $Q(x) = \sum_{j=1}^6 q_j x^{j-1}$ . Note that  $p_6 > 0$  and  $q_6 > 0$ , so  $P$  and  $Q$  are real polynomials of degree 5; in particular, each has at least one real root. Now, the equidistribution of the sum is equivalent to the relation

$$11 P(x)Q(x) = 1 + x + \cdots + x^{10} = (x^{11} - 1)/(x - 1).$$

However, this identity cannot hold, since the (cyclotomic) polynomial on the right has no real roots.



## 12. An Unexpected Inequality

Let  $a_1, a_2, \dots$  be positive reals. Then

$$\limsup_{n \rightarrow \infty} \left( \frac{1 + a_{n+1}}{a_n} \right)^n \geq e.$$

Furthermore, the inequality need not hold with  $e$  replaced by a larger number.

*Proof.* Suppose that the inequality does not hold. Then for some  $0 < c < 1$  there is an  $n_0$  such that

$$\frac{1 + a_{n+1}}{a_n} < 1 + \frac{c}{n}$$

whenever  $n \geq n_0$ , i.e.,

$$a_{n+1} < \left(1 + \frac{c}{n}\right) a_n - 1.$$

Consequently, by induction on  $k$ ,

$$a_{n_0+k} < a_{n_0} \prod_{i=0}^{k-1} \left(1 + \frac{c}{n_0 + i}\right) - k$$

for  $k = 1, 2, \dots$ , and so

$$a_{n_0+k} < a_{n_0} \exp \left( \sum_{i=0}^k \frac{c}{n_0 + i} \right) - k < a_{n_0} (n_0 + k)^c - k.$$

However, for  $k$  large enough,  $a_{n_0} (n_0 + k)^c - k < 0$ . This contradicts the assumption that  $a_{n_0+k}$  is positive, and so our inequality is proved.

To see that the inequality is best possible, let  $K > 1$ , and set  $a_n = Kn$ ,  $n = 1, 2, \dots$ . Then

$$\lim_{n \rightarrow \infty} \left( \frac{1 + a_{n+1}}{a_n} \right)^n = \lim_{n \rightarrow \infty} \left( \frac{1 + K(n+1)}{Kn} \right)^n = \left( 1 + \frac{1+1/K}{n} \right)^n = e^{1+1/K},$$

showing that  $e$  cannot be replaced by a number greater than  $e^{1+1/K}$ . Letting  $K \rightarrow \infty$ , we find that  $e$  cannot be replaced by any greater number.  $\square$

**Notes.** The problem is from *Littlewood's Miscellany*, ed. B. Bollobás (Cambridge University Press). Littlewood himself attributed the problem to George Pólya.

### 13. Colouring Lines: the Erdős–Selfridge Theorem

*In the Maker–Breaker game, with Maker wishing to construct a  $k$ -point line and Breaker starting, if there are  $2^k - 1$  lines then Breaker has a winning strategy, but for some arrangement of  $2^k$  lines Maker has a winning strategy. If Maker starts, then she wins on some collections of  $2^{k-1}$  lines of  $k$  points each, but loses whenever there are fewer than  $2^{k-1}$  lines.*

*Proof.* This beautiful and simple result is due to Erdős and Selfridge. The main assertion is the first: the others are essentially trivial.

As in the problem, Maker colours the points red and Breaker colours them blue. Maker wins if she can produce an entirely red line.

Suppose then that we have  $2^k - 1$  lines, each with  $k$  points. As the points get coloured, the lines change. At each stage, a line containing a point coloured blue is taken out of the set, and a line with some of its points coloured red and none coloured blue gets shortened by removing its red points. Maker wins if she can make the empty set a line.

Let us look at the set of current lines after each pair of moves by Breaker and Maker. Assign each current line of  $t$  points weight  $2^{-t}$ . Note that at the beginning the total weight is less than 1, and after a pair of Breaker–Maker moves, assuming that Breaker chooses a point that reduces the weight as much as possible, the total weight does not increase. Indeed, let the weight of a point be the sum of the weights of the lines containing it, and let Breaker pick a point  $x$  of maximal weight, say, weight  $w(x)$ . In the next move Maker picks a point  $y$  whose weight before Breaker’s last move was  $w(y) \leq w(x)$ . Maker’s move doubles the weight of each current line containing  $y$  but not  $x$ , so that after a pair of Breaker–Maker moves the total weight decreases by at least  $w(x) - w(y) \geq 0$ . Since at the start of the game the total weight of all the lines was less than 1, the empty set can never become a line, so Maker cannot construct a red line.

To see that this bound is best possible, let  $\mathcal{L}$  be the set of  $2^k$   $k$ -subsets  $\{c_1, c_2, \dots, c_k\}$  of the  $2k$ -set  $\{a_1, b_1, a_2, b_2, \dots, a_k, b_k\}$  such that each  $c_i$  is either  $a_i$  or  $b_i$ . To win the game, all Maker has to do is counter  $a_i$  by  $b_i$  and  $b_i$  by  $a_i$ .

If Maker starts, then she wins on some collections of  $2^{k-1}$  lines (each of  $k$  points, namely on the set of lines obtained from the construction above by identifying  $a_1$  and  $b_1$ , but loses whenever there are fewer than  $2^{k-1}$  lines. Indeed, at best, Maker’s first move ‘shortens’ all lines by 1, so we are done by the main assertion.  $\square$

**Notes.** As noted by Erdős and Selfridge, the argument above gives a somewhat more general result. Let  $\mathcal{L}$  be a set of lines such that  $\sum_{L \in \mathcal{L}} 2^{-|L|} < 1$ , where the lines need not have the same number of points. Then Breaker has a winning strategy for the game she starts.

The Erdős–Selfridge theorem was inspired by some results of Hales and Jewett about Tic–Tac–Toe in an  $n$ -dimensional grid of side  $k$ . In that setup there are  $\frac{1}{2}((k+2)^n - k^n)$  winning lines, so the result above shows that Breaker has a winning strategy if  $k > c \log n$ . For a huge body of exciting results in this area, see the beautiful survey paper by Beck.

J. Beck, Positional games, *Combin., Prob. and Comput.* **14** (2005), 649–696.

P. Erdős and J.L. Selfridge, On a combinatorial game, *J. Combin. Theory Ser. B* **14** (1973), 298–301.

A.W. Hales and R.I. Jewett, Regularity and positional games, *Trans. Amer. Math. Soc.* **106** (1963), 222–229.

## 14. Independent Sets

Let  $G$  be a graph with vertex set  $V$ , and write  $d(v)$  for the degree of a vertex  $v$ . Then  $G$  contains at least

$$i(G) = \sum_{v \in V} \frac{1}{d(v) + 1}$$

independent vertices.

*Proof.* We shall prove by induction on the number of vertices that the greedy algorithm described in the *Hint* does construct at least  $i(G)$  independent vertices. As this is obvious if there is only one vertex, let us turn to the proof of the induction step. Let  $x$  be a vertex of minimal degree, with neighbours  $y_1, \dots, y_k$ , and let  $H$  be the graph obtained from  $G$  by deleting the vertices  $x, y_1, \dots, y_k$ . By the induction hypothesis, our algorithm constructs at least  $i(H)$  vertices of  $H$ , so it constructs at least  $i(H) + 1$  vertices altogether. Hence, to complete the proof all we have to show is that  $i(G) \leq i(H) + 1$ . To see this note that every  $y_i$  has degree at least  $k$ , the degree of  $x$ , and the degree of a vertex in  $H$  is at most as large as its degree in  $G$ , so  $i(G) - i(H)$  is

$$\begin{aligned} \frac{1}{d(x) + 1} + \sum_{i=1}^k \frac{1}{d(y_i) + 1} + \sum_{z \in V(H)} \left( \frac{1}{d_G(z) + 1} - \frac{1}{d_H(z) + 1} \right) \\ \leq \frac{1}{k + 1} + \sum_{i=1}^k \frac{1}{k + 1} = 1, \end{aligned}$$

as required. □

**Notes.** This result was proved independently by Caro and Wei, and appeared in technical reports: Y. Caro, New results on the independence number, Technical Report, University of Tel Aviv, Israel, 1979, and V.K. Wei, A lower bound on the stability number of a simple graph, Technical Memorandum TM 81-11217-9, Bell Laboratories, 1981. It seems that the first journal publication mentioning these results was Y. Caro and Z. Tuza, Improved lower bounds on  $k$ -independence, *J. Graph Theory* **15** (1991), 99–107.

## 15. Expansion into Sums $2^i 3^j$

*Every natural number  $n$  is the sum of numbers of the form  $2^i 3^j$  such that no summand divides the other.*

*Proof.* Let us apply induction on  $n$ . For  $n = 1$  there is nothing to prove, so suppose that  $n > 1$  and the assertion holds for smaller values of  $n$ . If  $n$  is even then write  $n/2$  as a suitable sum  $n/2 = s_1 + \cdots + s_\ell$ , and note that  $n = 2s_1 + \cdots + 2s_\ell$  will do. If  $n$  is odd, choose  $k$  such that  $3^k \leq n < 3^{k+1}$ . Then  $n - 3^k$  is even. By the induction hypothesis,  $n' = (n - 3^k)/2$  can be written as a suitable sum  $n' = s_1 + \cdots + s_\ell$ ; then  $n = 2s_1 + \cdots + s_\ell + 3^k$  will do since  $s_i < 3^k$  for every  $i$ , so no  $s_i$  is divisible by  $3^k$ .

**Notes.** This assertion is due to Paul Erdős; I was reminded of it by Cecil Rousseau.

□

## 16. A Tennis Match

*Let  $p$  be the probability that  $A$  (Alice) wins her serve, and  $q$  the probability that  $B$  (Bob) wins his. Then the probability that  $A$  is the first to reach  $n$  wins is independent of the scheme ('alternating serve' AS or 'winner serves' WS) employed.*

*Proof.* We need a crucial observation: if  $A$  wins the set (and so the match) in the WS scheme then she serves exactly  $n$  times. Indeed,  $A$  certainly wins the last game, as she wins the set, and by agreement she serves in the first game. In every game after the first, she serves if and only if she won the previous game. Therefore, as altogether she wins  $n$  games, she serves exactly  $n$  times.

Now, as suggested in the *Hint*, let  $A$  and  $B$  continue the play for  $2n - 1$  games, and note that the winner is the one who wins at least  $n$  of these  $2n - 1$  games, no matter what happens in the remaining games; *in particular, no matter what serving scheme is employed*. When playing the AS scheme, we let them continue in that fashion for a total of  $2n - 1$  games so that when they finish,  $A$  will have served in  $n$  games and  $B$  in  $n - 1$  games. When playing the WS scheme, we let  $B$  serve in the rest of the games (until a total of  $2n - 1$  games are played). Since in the WS scheme at the time  $A$  wins her  $n$ th game, she has served exactly  $n$  times, we find again that when all  $2n - 1$  games are finished,  $A$  will have served  $n$  times and  $B$  exactly  $n - 1$  times. Hence, in either case,  $A$  and  $B$  play  $2n - 1$  games, with  $A$  serving in  $n$  of those games and  $B$  in  $n - 1$ . Since the probability of a win depends *only* on the server, the probability that Alice beats Bob is the same, whether they play according to the AS scheme or the WS scheme.  $\square$

## 17. A Triangle Inequality: Another Erdős Problem for Epsilons

Let  $T$  be a triangle with largest angle no more than  $\pi/2$ . Let  $R$  be the circumradius of  $T$ ,  $r$  its in-radius and  $h$  the length of its longest height. Then  $r + R \leq h$ . Furthermore, equality holds if and only if  $T$  is either a regular triangle or an isosceles right-angled triangle.

*Proof.* It is easily checked that equality holds for a regular triangle and an isosceles right-angled triangle. Thus our task is to show that we have strict inequality in every other case.

Let  $ABC$  be our triangle  $T$  and  $O$  its circumcentre; suppose that  $AB \geq AC \geq BC$ , so that the angle  $\alpha$  at  $A$  is smallest, and the angle  $\gamma$  at  $C$  is largest but not more than  $\pi/2$ . Note that the longest altitude,  $h$ , is the distance from  $A$  to  $BC$ . Let  $B'$  be the point on the half-line  $CB$  with  $CB' = CA$ . If  $AC > BC$  then, replacing  $ABC$  by the isosceles triangle  $AB'C$ ,  $h$  does not change, while  $r$  and  $R$  strictly increase, so in proving the result we may assume that  $AC = BC$ .

From here on, the problem can be solved by simple (and rather boring) analysis; still, to satisfy the interested reader, we quickly give the details of one of the many possible solutions.

Clearly, we may assume that  $R = 1$ . The angle of the isosceles triangle  $AOB$  at  $O$  is  $2\gamma$ , so  $c = AB = 2 \sin \gamma$ ; similarly,  $a = BC = b = AC = 2 \sin \alpha$ , so the perimeter length of our triangle is  $p = 4 \sin \alpha + 2 \sin \gamma$  (see Figure 31).

The altitude from  $C$  is  $m = 1 + \cos \gamma$ , and so the area of  $ABC$  is  $cm/2 = \sin \gamma(1 + \cos \gamma)$ . Consequently, the altitude of our triangle from  $A$  is

$$h = \frac{cm}{2} = \frac{\sin \gamma(1 + \cos \gamma)}{\sin \alpha},$$

and the in-radius is

$$r = \frac{cm}{p} = \frac{\sin \gamma(1 + \cos \gamma)}{2 \sin \alpha + \sin \gamma}.$$

Our task is to show that  $h - r \geq 1$  if  $\pi/4 \leq \alpha \leq \pi/3$ . Since  $2\alpha + \gamma = \pi$ , we have  $\sin \gamma = \sin 2\alpha$  and  $\cos \gamma = -\cos 2\alpha$ , so

$$\begin{aligned} h - r &= \frac{\sin \gamma(1 + \cos \gamma)}{\sin \alpha} - \frac{\sin \gamma(1 + \cos \gamma)}{2 \sin \alpha + \sin \gamma} \\ &= \frac{\sin 2\alpha(1 - \cos 2\alpha)}{\sin \alpha} - \frac{\sin 2\alpha(1 - \cos 2\alpha)}{2 \sin \alpha + \sin 2\alpha} \\ &= 4 \cos \alpha(1 - \cos^2 \alpha) - 2 \cos \alpha(1 - \cos \alpha). \end{aligned}$$

Writing  $t$  for  $\cos \alpha$ , we have to show that  $f(t) = -4t^3 + 2t^2 + 2t > 1$  for  $1/2 = \cos(\pi/3) < t < \sqrt{2}/2 = \cos(\pi/4)$ .

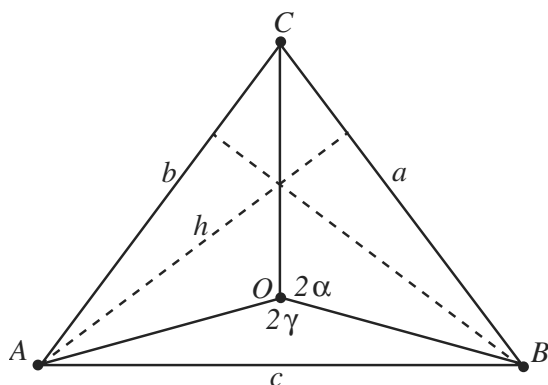


Fig. 31. The isosceles triangle  $ABC$ :  $a = BC = b = AC = 2 \sin \alpha$ ,  $c = AB = 2 \sin \gamma$ , the altitude from  $C$  is  $m = 1 + \cos \gamma$ , and from  $A$  (and  $B$ ) is  $h$ .

Now,  $f'(t) = -12t^2 + 4t$  and  $f''(t) = -24t + 4 < 0$  in the range  $1/2 \leq t \leq \sqrt{2}/2$ , so  $f(t)$  is a strictly concave function on this interval. As  $f(1/2) = f(\sqrt{2}/2) = 1$ , we have  $f(t) > 1$  for every  $t$  satisfying  $1/2 \leq t \leq \sqrt{2}/2$ , as required.  $\square$



## 18. Planar Domains of Diameter 1

- (i) Every planar domain  $D$  of diameter 1 is contained in a regular hexagon of width 1, i.e., side-length  $\sqrt{3}/3$ .
- (ii) Every planar domain of diameter 1 can be partitioned into three sets of diameter at most  $\sqrt{3}/2$ .

*Proof.* (i) Given a direction  $\ell$ , there is a strip of width 1 in the direction of  $\ell$  that contains  $D$ . Hence,  $D$  is contained in a hexagon  $H_\ell$  such that each angle is  $2\pi/3$ , any two opposite sides are at distance 1, and some two of its sides are parallel to  $\ell$ . Let  $a_1, \dots, a_6$  be the lengths of the sides of  $H_\ell$ . Then, as the angles are  $2\pi/3$ , we have  $a_1 + a_2 = \sqrt{3}$ ,  $a_2 + a_3 = \sqrt{3}$ ,  $\dots$ ,  $a_6 + a_1 = \sqrt{3}$ . In particular,  $a_1 = a_3 = a_5$  and  $a_2 = a_4 = a_6$ .

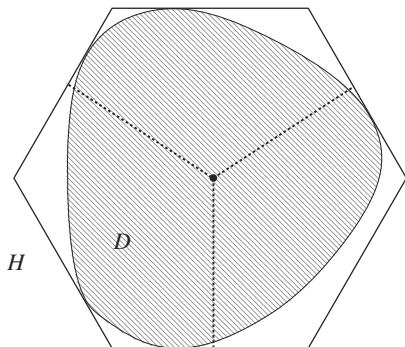


Fig. 32. A regular hexagon  $H$  of width 1 containing a domain  $D$  of diameter 1, and the division of  $H$  into three parts.

Let us choose  $H_\ell$  to be a continuous function of the direction  $\ell$ ; this is easily done. Then each  $a_i(\ell)$  is a continuous function of  $\ell$ . If  $\ell'$  is a line that forms an angle  $\pi/3$  with  $\ell$  then  $a_1(\ell') = a_2(\ell)$  and  $a_2(\ell') = a_1(\ell)$ . Consequently, if  $a_1(\ell) < a_2(\ell)$ , say, then  $a_1(\ell') > a_2(\ell')$ . Hence, as the functions  $a_i(\ell)$  are continuous, during the rotation of  $\ell$  into  $\ell'$  (through angle  $\pi/3$ ) we encounter a direction  $\ell''$  with  $a_1(\ell'') = a_2(\ell'')$ . But then  $H_{\ell''}$  is a regular hexagon of width 1 (and side-length  $\sqrt{3}/2$ ) containing  $D$ .

(ii) Let  $H$  be a regular hexagon of width 1 containing  $D$  whose existence is guaranteed by part (i). Cut  $H$  into three congruent pentagons by dropping perpendiculars on alternating sides from the centre of  $H$ , as in Figure 32. Each of these pentagons has diameter  $\sqrt{3}/2$ . □

**Notes.** This is the first non-trivial (not entirely trivial?) case of the Borsuk problem, to be discussed in Problems 42 and 133.

## 19. Orienting Graphs

*For every graph there is an orientation of the edges such that for every vertex the difference of the out-degree and in-degree is at most 1.*

*Proof.* Repeat the following operation while you can: find a cycle, orient it cyclically, and delete it from the graph. When we have no more cycles, our current graph is a forest, i.e., a graph whose every component is a tree. Thus it suffices to solve our problem for a tree (rather than a general graph). For a tree, orient the edges starting from a root, always making sure that the in-degree and out-degree do not differ by more than 1. This can be done in several natural ways.

For example, given a tree  $T$ , pick a vertex  $x_0 \in V(T)$  and orient (all) the edges incident with  $x_0$  one by one, sending the first out, the second in, etc. Next, pick a vertex  $x_1$  incident with one of the edges oriented so far and starting with that edge (and keeping its orientation), orient (all) the edges incident with  $x_1$  alternately in and out. Repeat this procedure, always picking a vertex incident with an oriented edge. Note that we shall never get to a vertex incident with two oriented edges before we orient the remaining edges incident with it.

Another trivial method goes as follows. Starting with a tree without oriented edges, pick a longest path consisting of *unoriented edges* and orient its edges ‘the same way’. Note that at the end of the procedure, every vertex  $x$  is the end-vertex of at most one path, since two paths ending in  $x$  would have been concatenated. Hence, the out-degree and in-degree of  $x$  differ by at most 1.  $\square$

## 20. A Simple Clock

*How many times a day is it impossible to tell the time by a clock with identical hour and minute hands, provided we can always tell whether it is a.m. or p.m.?*

*Solution.* Consider two clocks: one running normally, and the other twelve times as fast. Thus the hour hand of the fast clock always points at the same number as the minute hand of the accurate one. We cannot tell the time by looking at our clock with identical hands whenever these two clocks show the same time, *and the two hands do not coincide*. Now, from just after noon till midnight the minute hand of the fast clock completes 144 revolutions, while the hour hand of the accurate clock goes round just once, so the two clocks show the same time 143 times. During this period the two hands of the accurate clock coincide 11 times, starting with 60/11 minutes past one o'clock, so from noon till midnight we cannot tell the time  $143 - 11 = 132$  times. Hence the answer to our question is 264. This is likely to be considerably larger than most people would guess when pressed for a snap answer.

□

**Notes.** This problem is from the paper *A clock puzzle* by Andy Latta in *Puzzlers' Tribute – A Feast for the Mind*, D. Wolfe and T. Rodgers (eds), A.K. Peters (2002), xvi + 420 pp., see pp. 41–42.

## 21. Neighbours in a Matrix

*Every  $n \times n$  matrix whose entries are  $1, 2, \dots, n^2$  in some order has two neighbouring entries (in a row or in a column) that differ by at least  $n$ .*

*Proof.* Let us call a row or column of our matrix a *line*. Let  $m$  be the smallest integer such that some line is filled with numbers at most  $m$ . We may assume that this line is the first row and  $m$  is in the  $(1, 1)$  position of this row. Since none of columns two, three,  $\dots, n$ , is filled with the numbers  $1, 2, \dots, m$ , every column, with the possible exception of the first, has two neighbouring entries, one of which is at most  $m - 1$  and the other at least  $m + 1$ . Consequently, the difference of at least one of these  $n - 1$  pairs is at least  $n$ : indeed, the maximum of the  $n - 1$  numbers in these pairs that are greater than  $m$  is at least  $m + n - 1$ , and the other number in this pair is at most  $m - 1$ .  $\square$



## 22. Separately Continuous Functions

Let  $f : S = [0, 1]^2$  be separately continuous in its variables and taking 0 on a dense subset of  $S$ . Then  $f$  is identically 0.

*Proof.* Suppose for a contradiction that  $f$  is not identically 0. Then we may assume that  $f(x_0, y_0) \geq 3$ , where  $0 < x_0 < 1$  and  $0 < y_0 < 1$ .

Since  $f(x, y_0)$  is a continuous function of  $x$ , there is an interval  $I = [a, b]$ ,  $0 \leq a < b \leq 1$ , such that  $f(x, y_0) \geq 2$  for  $x \in I$ .

Now, for  $n \geq \min\{y_0, 1 - y_0\}^{-1}$ , set

$$F_n = \{x \in I : f(x, y_0 + y) \geq 1 \text{ if } |y| \leq 1/n\}.$$

As  $f$  is continuous in  $x$ , each  $F_n$  is a closed subset of  $I$ . Furthermore, since  $f$  is continuous in  $y$  and  $f(x, y_0) \geq 2 > 1$  for every  $x \in I$ , we have  $I = \bigcup_n F_n$ . By the Baire category theorem (see, e.g., p. 76 of B. Bollobás, *Linear Analysis*, Cambridge University Press, 1990, xi+240 pp.), some  $F_n$  has a non-empty interior, i.e., contains an interval  $I_n = [c, d]$  with  $0 \leq c < d \leq 1$ . Consequently,  $f(x, y) \geq 1$  on the rectangle  $(x, y) \in I_n \times [y_0 - 1/n, y_0 + 1/n] \subset S$ , contradicting the assumption that  $f^{-1}(0)$  is dense in  $S$ .  $\square$

**Notes.** This was one of my favourite exercises for the *Linear Analysis* course in Cambridge in the early 1970s.

## 23. Boundary Cubes

*The maximum of the number of integral unit cubes meeting the boundary of a down-set  $D$  in the solid  $d$ -dimensional cube  $[0, n]^d$  is  $n^d - (n - 1)^d$ .*

*Proof.* To see that we can have  $n^d - (n - 1)^d$  boundary cubes, simply take

$$D = \{\mathbf{x} = (x_i)_1^n : 0 \leq x_i \leq n - 1/2 \text{ for every } i\}.$$

Turning to the real assertion, that we cannot have more than  $n^d - (n - 1)^d$  boundary cubes, let  $\mathcal{C}$  be the set of boundary cubes, i.e., integral unit cubes meeting both a down-set and its complement. Note that if  $C = \prod_{i=1}^d [c_i - 1, c_i]$  and  $C' = \prod_{i=1}^d [c'_i - 1, c'_i]$  are distinct integral unit cubes in  $\mathcal{C}$  then we cannot have  $c_i \leq c'_i$  for every  $i$ . Consequently, if for  $C = \prod_{i=1}^d [c_i - 1, c_i] \in \mathcal{C}$  we set  $\varphi(C) = (c_1 + r, c_2 + r, \dots, c_d + r)$ , where  $r = n - \max c_i$ , then  $\varphi$  is a one-to-one map. Clearly,  $\varphi$  maps  $\mathcal{C}$  into the set of lattice points  $(a_1, \dots, a_d)$  with every  $a_i$  an integer and  $1 \leq \min a_i \leq \max a_i = n$ . Since there are  $n^d - (n - 1)^d$  such lattice points,  $|\mathcal{C}| \leq n^d - (n - 1)^d$ . □

## 24. Lozenge Tilings

*Every tiling of a regular hexagon by congruent lozenges contains the same number of lozenges of each orientation.*

*Proof.* Consider a tiling  $\tau$  of the regular hexagon of side-length  $n$  by  $3n^2$  lozenges (rhombi with angles  $\pi/3$  and  $2\pi/3$ ), each of unit side-length. Note that each lozenge has one of three possible orientations: our task is to prove that  $\tau$  contains  $n^2$  of each kind, as in the *canonical tiling* shown in Figure 33. We shall use this figure to refer to the three kinds of tile as *right*, *left* and *bottom*, or, keeping the initials, as *red*, *lilac* and *blue*.

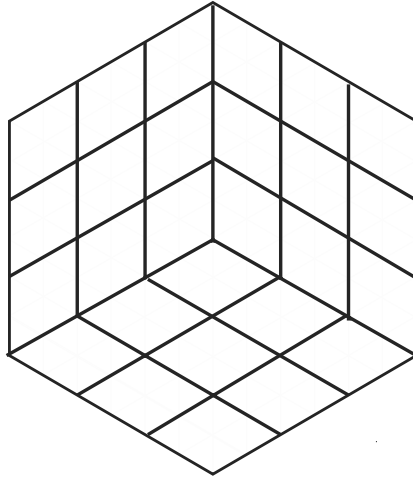


Fig. 33. The canonical tiling of  $H_3$  with 27 lozenges, of which 9 are red, 9 lilac, and 9 blue.

The first proof we give is entirely visual. Look at the tiling in Figure 34. What do you see? A family of unit cubes packed into an  $n \times n \times n$  box in a such a way that each unit cube goes as close as possible to the corner of the box where the bottom, right and left faces of the box meet. Looking at this packing from above, we see the floor of the box covered by the top faces of the cubes and the uncovered part of the floor, also tiled by blue tiles. Hence, there are  $n^2$  blue tiles. Analogously, there are  $n^2$  red tiles and  $n^2$  lilac ones, as claimed.

To turn this ‘proof by pictures’ into a proper mathematical proof we assign the coordinates  $(0, 0, n)$  to the top vertex of the hexagon (as in Figure 34), and extend this assignment to the vertices of the tiling such that a step down and left increases the  $x$  coordinate by 1, a step down and right increases the  $y$  coordinate by 1, and

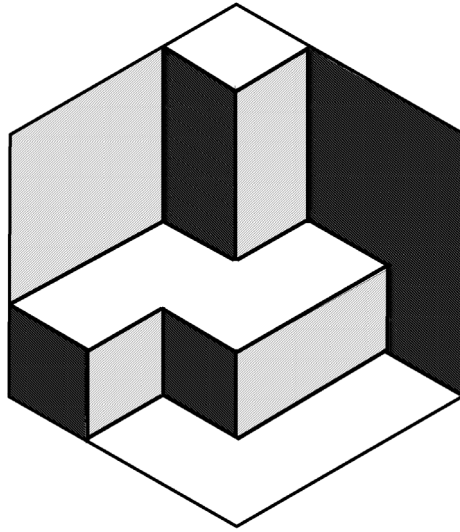


Fig. 34. A lozenge tiling – or a packing of unit cubes into a box.

a step straight down decreases the  $z$  coordinate by 1. If this assignment is possible then we have a packing of unit cubes into the  $n \times n \times n$  box  $[0, n]^3$ , as required. What we have to show is that such an assignment is possible, i.e., the rules above are consistent. This follows from the facts that the hexagon is contractible, and the rules give a consistent assignment to the vertices of a lozenge, no matter how it is oriented.

Let us give another proof of this result. There are two ways of tiling a regular hexagon of unit side with unit lozenges, one of each type; replacing the one in which the blue tile is at the top by the one in which it is at the bottom is a *down-flip*, as shown in Figure 35. Note that a down-flip is also a left-flip (bringing the lilac tile closer to the left side) and a right-flip (bringing the right tile closer to the right side), so we could call it just a *flip*.

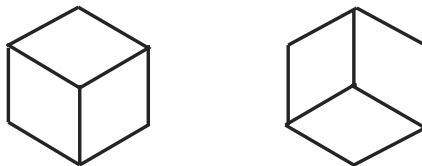


Fig. 35. A down-flip.



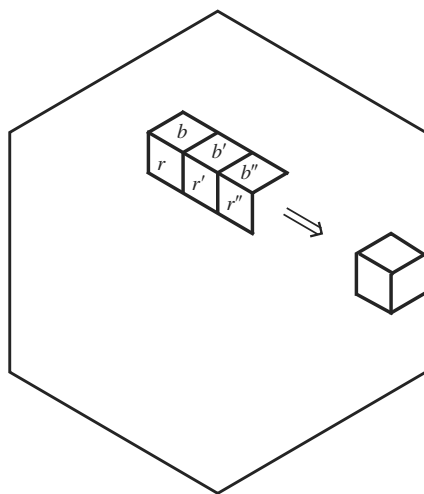


Fig. 36. A blue tile  $b$  and a red tile  $r$  touching one of its lower sides, implying the existence of  $b'$  and  $r'$ , and so on.

Starting from any tiling, after a number of down-flips we arrive at a tiling  $\tau$  in which no down-flip can be performed. We claim that  $\tau$  is the canonical tiling shown in Figure 33.

If this is not the case then, without loss of generality, there is a blue tile  $b$  such that at least one of the two tiles touching the lower edges of  $b$  is not a blue tile, but a red one,  $r$ , as in Figure 36.

There cannot be a lilac tile touching these tiles  $b$  and  $r$ , since then a flip could be performed in  $\tau$ , so their neighbours (downward) are again blue and red, as in Figure 36. This pair of tiles  $(b', r')$  is exactly as  $(b, r)$ , so it spawns another pair  $(b'', r'')$ . Continuing in this way, we find an infinite sequence of blue-red pairs. This contradiction completes our second proof.  $\square$

**Notes.** It seems that David and Tomei were the first to prove this beautiful and basic fact about tilings although, very modestly, they write that “*We believe that the result is known and proven, but we were unable to find any references*”. Many years earlier, MacMahon had considered similar problems, but it seems that he did not mention this basic fact. Since this result of David and Tomei, much research has been done on enumerating lozenge tilings, changing one into another, and finding ‘typical’ lozenge tilings; see the papers by Ciucu and Krattenthaler; Luby, Randall and Sinclair; and Ciucu, among many others.

David and Tomei called these tilings ‘calisson tilings’. *Calisson* is a delicious sweet from Aix-en-Provence: in the XVIIth century it was believed to protect

against the plague. It is a pity that this romantic name has been replaced by the much more prosaic and much less accurate ‘lozenge’.

M. Ciucu, A random tiling model for two dimensional electrostatics, *Memoirs of the American Mathematical Society* **178** (2005), x + 144 pp.

M. Ciucu and C. Krattenthaler, The number of centered lozenge tilings of a symmetric hexagon, *J. Combin. Theory Ser. A* **86** (1999), 103–126.

M. Ciucu and C. Krattenthaler, Enumeration of lozenge tilings of hexagons with cut-off corners, *J. Combin. Theory Ser. A* **100** (2002), 201–231.

G. David and C. Tomei, The problem of calissons, *Amer. Math. Monthly* **96** (1989), 429–431.

M. Luby, D. Randall and A. Sinclair, Markov chain algorithms for planar lattice structures, *SIAM J. Comput.* **31** (2001), 167–192 (electronic).

P.A. MacMahon, *Combinatory Analysis*, two volumes bound as one, Chelsea Publishing Co., (1960), xix + 302 and xix + 340 pp.

## 25. A Continuum Independent Set

To every real number  $x$ , assign a finite set  $\Phi(x) \subset \mathbb{R} \setminus \{x\}$ . Call a set  $S \subset \mathbb{R}$  independent if  $x \in \Phi(y)$  does not hold for  $x, y \in S$ , that is if  $S \cap \Phi(S) = \emptyset$ . Then there is an independent set  $S \subset \mathbb{R}$  whose cardinality is that of the continuum.

*Proof.* For every  $x \in \mathbb{R}$  there is an interval  $I(x) = [r_1(x), r_2(x)]$  with rational end-points such that  $x \in I(x)$  and  $I(x) \cap \Phi(x) = \emptyset$ . As there are only countably many intervals with rational end-points, for some interval  $J$  the cardinality of the set  $S = \{x \in \mathbb{R} : I(x) = J\}$  is that of the continuum. Since  $S \subset J$  and  $\Phi(S) \cap J = \emptyset$ , the set  $S$  is independent.  $\square$

**Notes.** This sweet little result is due to Dezső Lázár, a member of the famous *Anonymus Circle* Paul Erdős belonged to during his university years. Anonymus (spelt like this!) was the notary of the Hungarian king Béla III, 1172–1196, and the first historian of his country: his *Gesta Hungarorum* tells the story of how the *Magyars* occupied a land three times the area of present-day Hungary in 896 (the Hungarian 1066). His impressive statue, a landmark in Budapest, was the meeting place of Erdős and his fellow undergraduates, Esther Klein, Dezső Lázár, George Szekeres, and many others.

The problem Lázár solved arose from Turán's work on interpolation theory. Erdős communicated Lázár's proof to John von Neumann, back in Budapest for the summer, who liked it so much that he invited Lázár to submit a brief note to *Compositio Mathematica*.

D. Lázár, On a problem in the theory of aggregates, *Compositio Math.* (2) **3** (1936), 304.

## 26. Separating Families of Sets

*The minimal number of families needed to separate  $n$  elements is  $\lceil \log_2 n \rceil$ .*

*Proof.* Although this result is essentially trivial, it may be instructive to give several proofs differing mostly in their formulations.

Let  $S$  be a set of  $n$  elements, and let  $\mathcal{S} = \{S_1, \dots, S_k\}$  be a family of subsets of  $S$ . This family  $\mathcal{S}$  separates the elements of  $S$  if every element  $x$  can be identified by answering the following  $k$  questions: does  $x$  belong to  $S_i$ ,  $i = 1, \dots, k$ ? Now, there are  $2^k$  possible sequences of answers ('yes', 'yes', 'no', 'yes', 'no', etc.), and to identify all elements of  $S$  we need at least  $n$  different sequences of answers. Hence, if  $k$  sets separate the elements of  $S$  then  $2^k \geq n$ , i.e.,  $k \geq \lceil \log_2 n \rceil$ .

On the other hand, if  $2^k \geq n$  then  $k$  sets ('questions') suffice: simply identify the elements of  $S$  with (some of) the  $2^k$  yes-no sequences of length  $k$ .

At the risk of over-explaining this very simple proof, let us make it more formal, using the terminology of bipartite graphs. A graph  $B$  is bipartite if its vertex set is the union of two vertex classes,  $U_0$  and  $U_1$ , say, such that every edge of  $B$  joins a vertex in  $U_0$  to one in  $U_1$ . A bipartite graph with vertex classes  $U_0, U_1$  is *complete* if it contains all  $U_0 - U_1$  edges, i.e., if its edge set is the set of all  $|U_0||U_1|$  edges joining a vertex in  $U_0$  to one in  $U_1$ . We say that (the edges of) a graph  $G$  is *covered* by the graphs  $H_1, \dots, H_k$  if every edge of  $G$  belongs to at least one  $H_i$ .

The claim is then that the minimal number of complete bipartite graphs covering the complete graph  $K_n$  with  $n$  vertices is  $\lceil \log_2 n \rceil$ .

To prove this, first note that if  $n = 2^k$  then  $K_n$  can be covered with  $k$  bipartite graphs. Indeed, for  $i = 0, \dots, k-1$ , let  $B_i$  be the bipartite graph with vertex set  $V = \{0, 1, \dots, n-1\}$ , in which  $u$  is joined to  $v$  if in their dyadic expansions they differ in the  $i$ th digits. To spell it out, for  $u = \sum_{j=0}^{k-1} u_j 2^j$  and  $v = \sum_{j=0}^{k-1} v_j 2^j$  we join  $u$  to  $v$  by an edge if  $u_i \neq v_i$ . Equivalently, let  $B_i$  be the complete bipartite graph with vertex classes  $U_0^{(i)} = \{u \in V : \lfloor u/2^i \rfloor \text{ is even}\}$  and  $U_1^{(i)} = \{u \in V : \lfloor u/2^i \rfloor \text{ is odd}\}$ . Then the complete graph  $K_n$  with vertex set  $V$  is the union of the complete bipartite graphs  $B_i$ , so  $\log_2 n = k$  bipartite graphs suffice to cover a complete graph with  $n = 2^k$  vertices.

To prove the converse, we have to show that a complete graph on  $2^k + 1$  vertices cannot be covered by  $k$  bipartite graphs. A pedestrian solution goes by induction on  $k$  as follows. The assertion is obvious for  $k = 0$ . Suppose that  $k \geq 1$  and the assertion holds for smaller values of  $k$ . Let  $\bigcup_{i=1}^{\ell} B_i = K_n$ , where  $n = 2^k + 1$  and the  $B_i$  are bipartite graphs. Then at least  $\lceil n/2 \rceil = 2^{k-1} + 1$  of the vertices of  $K_n$  belong to the same class of  $B_{\ell}$ , so the complete graph on these vertices is covered by the  $\ell - 1$  bipartite graphs  $B_1, \dots, B_{\ell-1}$ . Consequently, by the induction hypothesis,  $\ell - 1 \geq k$ , and we are done.

A more elegant and no less formal way of putting these ideas goes as follows. Let  $K_n = \bigcup_{i=1}^k B_i$ , where each  $B_i$  is a bipartite graph, with vertex classes  $U_0^{(i)}$  and  $U_1^{(i)}$ , say. We may assume without loss of generality that  $U_0^{(i)} \cup U_1^{(i)} = V(K_n)$  for every  $i$ . For  $u \in V$ , let

$$\varphi_i(u) = \begin{cases} 0 & \text{if } u \in U_0^{(i)}, \\ 1 & \text{if } u \in U_1^{(i)}, \end{cases}$$

and define  $\varphi : V(K_n) \rightarrow \{0, 1\}^k$  by  $\varphi(u) = (\varphi_1(u), \dots, \varphi_k(u))$ . Then  $\varphi$  is a one-to-one map since if  $\varphi(u) = \varphi(v)$  then no  $B_i$  contains the edge  $uv$ . Hence  $n \leq 2^k$ . Conversely, if  $n \leq 2^k$  then any injection  $\varphi : V(K_n) \rightarrow \{0, 1\}^k$  defines a cover of  $K_n$  by  $k$  bipartite graphs: join  $u$  to  $v$  in the  $i$ th bipartite graph if  $\varphi_i(u) \neq \varphi_i(v)$ .  $\square$

## 27. Bipartite Covers of Complete Graphs

Let  $0 \leq \ell < 2^k$  and  $n = 2^k + \ell$  be natural numbers. Then the minimal number of vertices in a family of bipartite graphs covering all the edges of the complete graph  $K_n$  is  $nk + 2\ell$ .

*Proof.* Suppose that  $K_n = \bigcup_{i=1}^r B_i$ , where  $B_1, \dots, B_r$  are bipartite graphs, and  $B_i$  has vertex classes  $U_0^i$  and  $U_1^i$ . For notational simplicity, suppose that  $V(K_n) = V = [n]$ . Following the *Hint*, note that, for every 0-1 sequence  $\varepsilon = (\varepsilon_i)_{i=1}^r$ , the set

$$V_\varepsilon = V \setminus \bigcup_{i=1}^r U_{\varepsilon_i}^i$$

has at most one element, since if  $u$  and  $v$  were in this set then no  $B_i$  would contain the edge  $uv$ .

Now, pick a random 0-1 sequence  $\varepsilon = (\varepsilon_i)_{i=1}^r$ , and consider the expectation  $\mathbb{E}(|V_\varepsilon|)$ . Since  $|V_\varepsilon| \leq 1$  for every  $\varepsilon$ , this expectation is at most 1. On the other hand, it is easy to give an explicit formula for  $\mathbb{E}(|V_\varepsilon|)$ . Indeed, for  $j \in [n] = V$ , let  $m_j$  be the number of bipartite graphs  $B_i$  containing  $j$  (the ‘multiplicity’ of covering vertex  $j$ ). Then the probability that this vertex  $j$  is not deleted from  $[n] = V$  when we take  $V_\varepsilon$  is exactly  $2^{-m_j}$ ; hence,

$$\mathbb{E}(|V_\varepsilon|) = \sum_{j=1}^n 2^{-m_j} \leq 1.$$

Finally, if  $B_i$  has  $b_i$  vertices then  $b = \sum_{i=1}^r b_i = \sum_{j=1}^n m_j$ , so the convexity of  $2^x$  implies that

$$1 \geq \mathbb{E}(|V_\varepsilon|) = \sum_{j=1}^n 2^{-m_j} \geq n 2^{-b/n}.$$

Consequently,  $2^{b/n} \geq n$ , i.e.,  $b = \sum_{i=1}^r b_i \geq n \log_2 n$ , as claimed.

To sharpen this bound, note that we may assume that the  $m_j$  take at most two values, and that these are  $k$  and  $k+1$ . If  $m_j = k$  for  $\alpha$  of the vertices  $j$ , then

$$\alpha 2^{-k} + (n - \alpha) 2^{-(k+1)} \leq 1,$$

and so

$$(n + \alpha) 2^{-(k+1)} \leq 1.$$

Consequently,  $\alpha \leq 2^k - \ell$ , and so

$$\sum_{i=1}^N |U_0^i \cup U_1^i| = \sum m_j = \alpha k + (n - \alpha)(k + 1) = nk + n - \alpha \geq nk + n - 2^k + \ell,$$

which is equal to  $nk + 2\ell$ .

To see that for  $n = 2^k + \ell < 2^{k+1}$  the bound  $nk + 2\ell$  can be attained, partition  $[n]$  into  $2^k - \ell$  sets of size 1 and  $\ell$  pairs. As in Problem 26, we can cover the edges between these  $2^k$  sets with  $k$  complete bipartite graphs (with  $n$  vertices each), and it is trivial to cover the  $\ell$  remaining edges with  $\ell$  bipartite graphs with two vertices each. Then summing the orders of the graphs gives a total of  $nk + 2\ell$ .

In conclusion, let us reformulate this result in terms of separation. Let  $S$  be a set with  $n$  elements. A *separating system of pairs on  $S$*  is a collection  $(S_1, T_1), \dots, (S_N, T_N)$  of disjoint pairs of subsets of  $S$  such that for every  $x, y \in S$  with  $x \neq y$  there is an index  $i$  with  $x \in S_i$  and  $y \in T_i$ , or  $x \in T_i$  and  $y \in S_i$ . Let  $S(n)$  be the minimum of the total number of vertices in the sets  $S_i, T_i$ :

$$S(n) = \min \left\{ \sum_{i=1}^N |S_i \cup T_i| : (S_i, T_i)_{i=1}^N \text{ is a separating system on } S \right\},$$

Then  $S(n) = nk + 2\ell$  for  $n = 2^k + \ell < 2^{k+1}$ . □

**Notes.** The result we have just proved is a considerable extension of the simple result (observation?) in Problem 26, since if  $k$  subgraphs of  $K_n$  have altogether at least  $n \log_2 n$  vertices then  $k \geq \log_2 n$ , and so  $k \geq \lceil \log_2 n \rceil$ . This extension was first proved by Hansel; a little later it was rediscovered by Katona and Szemerédi. The sharp bound in the question was observed by Bollobás and Scott.

B. Bollobás and A. Scott, On separating systems, *Europ. J. Combin.*, to appear.

Hansel, G., Nombre minimal de contacts de fermeture nécessaires pour réaliser une fonction booléenne symétrique de  $n$  variables, *C. R. Acad. Sci. Paris* **258** (1964), 6037–6040.

Katona, G., and Szemerédi, E., On a problem of graph theory, *Studia Sci. Math. Hungar* **2** (1967), 23–28.

## 28. Convexity and Intersecting Simplices: the Theorems of Radon and Carathéodory

- (i) Let  $X = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n+2}\}$  be a set of  $n + 2$  points in  $\mathbb{R}^n$  and, for a non-empty subset  $I$  of  $\{1, \dots, n + 2\}$ , let  $X(I)$  be the convex hull of the points  $\mathbf{x}_i$ ,  $i \in I$ . Then there are disjoint sets  $I, J$  such that  $X(I) \cap X(J) \neq \emptyset$ .
- (ii) The convex hull of a set  $X \subset \mathbb{R}^n$  is

$$\text{conv} X = \left\{ \sum_{i=1}^{n+1} \lambda_i \mathbf{x}_i : \mathbf{x}_i \in X, \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1 \right\}.$$

*Proof.* (i) This basic (and very simple) result in convex geometry is *Radon's theorem*.

By translating  $X$ , if necessary, we may assume that  $\mathbf{x}_{n+2} = 0$ . The set  $X' = X \setminus \{\mathbf{x}_{n+2}\}$  of  $n + 1$  vectors in the  $n$ -dimensional space  $\mathbb{R}^n$  is not independent, so there are constants  $\lambda_1, \dots, \lambda_{n+1}$ , not all 0, such that  $\sum_{i=1}^{n+1} \lambda_i \mathbf{x}_i = 0$ . Hence, setting  $\lambda_{n+2} = -\sum_{i=1}^{n+1} \lambda_i$ , we have  $\sum_{i=1}^{n+2} \lambda_i \mathbf{x}_i = 0$  and  $\sum_{i=1}^{n+2} \lambda_i = 0$ , with not all  $\lambda_i$  equal to 0.

Consequently, defining  $I = \{i : \lambda_i > 0\}$  and  $J = \{j : \lambda_j < 0\}$ , we have  $\sum_{i \in I} \lambda_i \mathbf{x}_i = \sum_{j \in J} \lambda_j \mathbf{x}_j$  and  $\sum_{i \in I} \lambda_i = -\sum_{j \in J} \lambda_j = \lambda > 0$ . Therefore the vector  $\sum_{i \in I} (\lambda_i / \lambda) \mathbf{x}_i = \sum_{j \in J} (-\lambda_j / \lambda) \mathbf{x}_j$  is in  $X(I) \cap X(J)$ .

(ii) By definition, the convex hull  $\text{conv} X$  of  $X$  is the smallest convex set containing  $X$ , i.e., the intersection of all convex sets containing it; trivially, it is also the set

$$\text{conv} X = \left\{ \sum_{i=1}^m \lambda_i \mathbf{x}_i : \mathbf{x}_i \in X, \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1, m = 1, 2, \dots \right\}.$$

What the basic result in this question, *Carathéodory's theorem*, tells us is that, in this identity, it suffices to take  $m = n + 1$  or, equivalently,  $m \leq n + 1$ .

To prove this assertion, write  $\mathbf{x} \in \text{conv} X$  as a convex linear combination  $\mathbf{x} = \sum_{i=1}^m \lambda_i \mathbf{x}_i$ ,  $\mathbf{x}_i \in X$ ,  $\lambda_i \geq 0$ ,  $\sum_{i=1}^m \lambda_i = 1$ , with  $m$  minimal. Suppose for a contradiction that  $m \geq n + 2$ . By part (i), Radon's theorem, there are disjoint non-empty sets  $I, J \subset \{1, \dots, m\}$  and strictly positive constants  $\mu_h$ ,  $h \in I \cup J$ , such that

$$\sum_{h \in I} \mu_h \mathbf{x}_h = \sum_{h \in J} \mu_h \mathbf{x}_h.$$

By renaming the points, if necessary, we may assume that  $I = \{1, \dots, k\}$ ,  $J = \{k + 1, \dots, \ell\}$  and  $t = \lambda_1 / \mu_1 = \max_{i \in I} \lambda_i / \mu_i$ . Then

$$\mathbf{x} = \sum_{i=1}^k (\lambda_i - t \mu_i) \mathbf{x}_i + \sum_{j=k+1}^{\ell} (\lambda_j + t \mu_j) \mathbf{x}_j + \sum_{h=\ell+1}^m \lambda_h \mathbf{x}_h = \sum_{i=2}^m v_i \mathbf{x}_i,$$

where  $v_i \geq 0$  and  $\sum_{i=2}^m v_i = 1$ . This contradiction completes the proof.  $\square$



**Notes.** The results in this Problem by Radon and Carathéodory, are among the first one encounters when studying convexity. It is customary to state Carathéodory's theorem in the following slightly more general form.

*If  $X$  is a subset of an affine  $n$ -dimensional plane in  $\mathbb{R}^N$ , then the convex hull of  $X$  is*

$$\text{conv}X = \left\{ \sum_{i=1}^{n+1} \lambda_i \mathbf{x}_i : \mathbf{x}_i \in X, \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1 \right\}.$$

Needless to say, the proof above applies to this form as well.

## 29. Intersecting Convex Sets: Helly's Theorem

Let  $\mathcal{C}$  be a finite family of convex sets in  $\mathbb{R}^n$  such that for  $k \leq n + 1$  any  $k$  members of  $\mathcal{C}$  have a non-empty intersection. Then the intersection of all members of  $\mathcal{C}$  is non-empty.

*Proof.* We may clearly assume that  $\mathcal{C}$  contains  $m \geq n + 1$  sets and the intersection of any  $n + 1$  members of it is non-empty. We shall prove by induction on  $m$  that the intersection of *all* members of  $\mathcal{C}$  is non-empty.

As for  $m = n + 1$  there is nothing to prove, let us turn to the induction step: let  $m \geq n + 2$  and suppose that the assertion holds for families with at most  $m - 1$  sets. Let  $\mathcal{C} = \{C_1, \dots, C_m\}$ . By the induction hypothesis we know that any  $m - 1$  members of  $\mathcal{C}$  have a point in common, say,

$$\mathbf{x}_i \in C_1 \cap C_2 \cap \dots \cap C_{i-1} \cap C_{i+1} \cap \dots \cap C_m$$

for  $i = 1, \dots, m$ . Putting it a little differently, if  $i \neq j$  then  $\mathbf{x}_i \in C_j$ .

As  $m \geq n + 2$ , by Radon's theorem, i.e., Part (i) of Problem 28, there are disjoint sets  $I, J \subset [m] = \{1, \dots, m\}$  such that

$$\mathbf{x} \in \text{conv}\{\mathbf{x}_i : i \in I\} \cap \text{conv}\{\mathbf{x}_j : j \in J\}$$

for some point  $\mathbf{x}$ . We may clearly assume that  $I$  and  $J$  partition  $[m]$ :  $I \cup J = [m]$ .

To check that this point  $\mathbf{x}$  is in every  $C_i$ , note that for each  $i \in I$  the point  $\mathbf{x}_i$  is in the convex set  $\bigcap_{j \in J} C_j$ , and so

$$\text{conv}\{\mathbf{x}_i : i \in I\} \subset \bigcap_{j \in J} C_j.$$

Analogously, we have

$$\text{conv}\{\mathbf{x}_j : j \in J\} \subset \bigcap_{i \in I} C_i.$$

Consequently,

$$\mathbf{x} \in \text{conv}\{\mathbf{x}_i : i \in I\} \cap \text{conv}\{\mathbf{x}_j : j \in J\} \subset \left( \bigcap_{j \in J} C_j \right) \cap \left( \bigcap_{i \in I} C_i \right) = \bigcap_{i=1}^m C_i,$$

completing the proof of the induction step.  $\square$

**Notes.** *Helly's theorem* is another fundamental result of convexity theory. This result can be used to prove numerous beautiful assertions in combinatorial geometry; in particular, we shall use it to answer Problem 127.

It is trivial that its word-for-word extension to infinite families is false: for example, with  $C_i = \{\mathbf{x} \in \mathbb{R}^n : x_i \geq i\}$ ,  $i = 1, \dots$ , any finite number of the sets  $C_i$  have

a non-empty intersection, but the intersection of all of them is empty. However, it is trivial that for *compact* convex sets we may take infinite families:

*if an infinite family of compact convex sets in  $\mathbb{R}^n$  is such that any  $n + 1$  members of the family have a non-empty intersection, then the intersection of all of them is non-empty.*

Helly first published his theorem in 1923, after earlier publications by Radon and König; his extension appeared in 1930. In 1963, Danzer, Grünbaum and Klee published a beautiful review article about the many uses of Helly's theorem; this is the article that established the result as one of the cornerstones of convexity theory.

L. Danzer, B. Grünbaum and V. Klee, Helly's theorem and its relatives, in *Convexity*, Proc. Symposia in Pure Maths, vol. VII., American Math. Soc., (1963), pp. 101–180.

E. Helly, Über Mengen konvexen Körper mit gemeinschaftlichen Punkten, *Jahresbericht der Deutscher Mathematiker Vereinigung* **32** (1923), 175–76.

E. Helly, Über Systeme von abgeschlossenen Mengen mit gemeinschaftlichen Punkten, *Monatsh. Math.* **37** (1930), 281–302.

### 30. Judicious Partitions of Points

Given  $n$  points,  $P_1, \dots, P_n$ , in the plane, there is a point  $P$  such that every line through  $P$  has at least  $n/3$  points  $P_i$  in each of the two closed half-planes it determines.

*Proof.* Let  $K$  be the convex hull of the  $P_i$ , and let  $\{H_\gamma : \gamma \in \Gamma\}$  be the collection of closed half-planes that contain more than  $2n/3$  of the points  $P_i$ . Then  $K_\gamma = H_\gamma \cap K$  is a compact convex set for each  $\gamma \in \Gamma$ . Clearly, any three members of the family  $\{K_\gamma : \gamma \in \Gamma\}$  intersect: indeed, as every  $K_\gamma$  contains *more than* two thirds of the points  $P_i$ , any three  $K_\gamma$  have at least one  $P_i$  in common. Hence, by Helly's theorem in Problem 29 (to be pedantic, by the trivial extension of Helly's theorem to infinite families of compact convex sets we noted there) there is a point  $P$  in  $\bigcap_{\gamma \in \Gamma} K_\gamma$ .

We claim that this point  $P$  will do. Indeed, let  $\ell$  be a line through  $P$ . Suppose for a contradiction that one of the two closed half-planes determined by  $\ell$ , the half-plane  $V$ , say, contains fewer than  $n/3$  of the  $P_i$ . Then the *open* complement  $U$  of this half-plane  $V$  contains *more than*  $2n/3$  of the  $P_i$ . Translating  $\ell$  by a sufficiently small amount, we find a closed half-plane  $W$  contained in  $U$  that contains *all* the  $P_i$  in  $U$ . In particular,  $W$  contains more than  $2n/3$  of the  $P_i$ , so it is one of the  $H_\gamma$ . Since  $P$  is not in  $W$ , we have found an  $H_\gamma$  that does not contain  $P$ . This contradiction completes our solution.  $\square$

**Notes.** The result above has the following trivial extension to  $d$  dimensions. Given  $n$  points in  $\mathbb{R}^d$ , there is a point  $P \in \mathbb{R}^d$  such that for every hyperplane  $H$  through  $P$  each of the two closed half-spaces determined by  $H$  contains at least  $dn/(d+1)$  of our points.

The bounds above ( $2n/3$  and  $dn/(d+1)$ ) are best possible: to see this, put the points close to the vertices of a regular  $d$ -dimensional simplex, placing about  $n/(d+1)$  into each group.

### 31. Further Lozenge Tilings

Let  $H(a)$  be a hexagon in which every angle is  $2\pi/3$  and the sequence of side-lengths is  $a = (a_1, \dots, a_6)$ , as in Figure 37. Then  $H(a)$  has a lozenge tiling if and only if  $a_1 = a_4$ ,  $a_2 = a_5$  and  $a_3 = a_6$ . If this holds then every lozenge tiling of  $H(a)$  has  $a_1 a_2$  bottom tiles,  $a_1 a_3$  right tiles and  $a_2 a_3$  left tiles, where  $a_1$  and  $a_2$  are the lengths of the bottom sides,  $a_3$  and  $a_1$  are the lengths of the right sides, and  $a_2$  and  $a_3$  are the lengths of the left sides.

*Proof.* Note first that if  $a_1 = a_4$ ,  $a_2 = a_5$  and  $a_3 = a_6$  then  $H(a)$  has a canonical tiling, as shown in Figure 38, with a central hexagon formed by three tiles, one of each kind.

To prove the main part of the assertion, that if there is a tiling then  $a_i = a_{i+3}$  for  $i = 1, 2, 3$ , we proceed as in the solution of Problem 24.

As there, starting from any tiling, after a number of down-flips we arrive at a tiling  $\tau$  in which no down-flip can be performed. Again, as proved there,  $\tau$  is such that, for any bottom lozenge, all the tiles below it are also bottom lozenges. Similarly, to the right of a right lozenge we have only right lozenges, and to the left of a left lozenge we have only left lozenges. Briefly, there is a central hexagon of unit side-length tiled with a bottom lozenge at its bottom, a right lozenge on the right and a left lozenge on the left. This implies that lozenges of the same kind tile

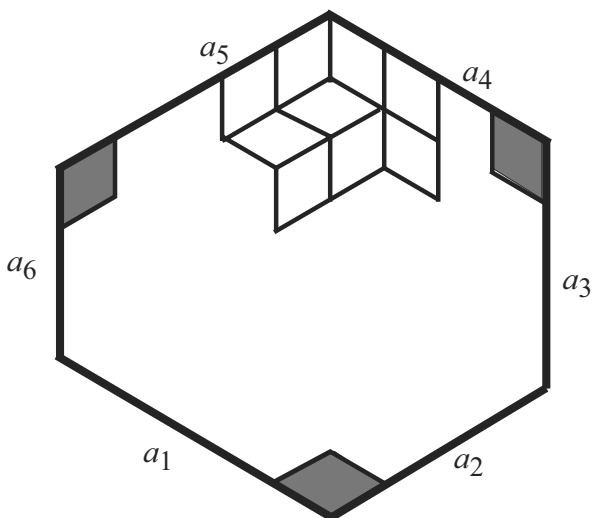


Fig. 37. A hexagon  $H(a)$  with side-lengths  $a_1, \dots, a_6$ , and the beginning of a lozenge tiling. A bottom tile is shaded, as are a left tile and a right tile.

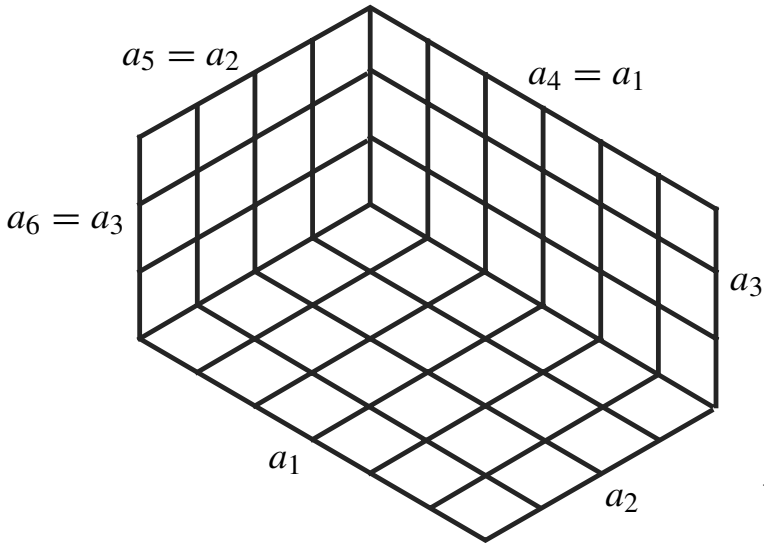


Fig. 38. The canonical lozenge tiling of a hexagon  $H(a)$  in which the opposite sides have equal lengths:  $a_1 = a_4 = 6$ ,  $a_2 = a_5 = 4$  and  $a_3 = a_6 = 3$ .

parallelograms and  $H(a)$  is the union of three of them, so the tiling is the canonical tiling shown in Figure 38, implying the assertions.  $\square$

**Notes.** Needless to say, another way of proving the assertion is to show that a lozenge tiling corresponds to a packing of unit cubes into the rectangular parallelepiped we can see in the canonical tiling, as in Figure 38.

### 32. Two Squares in a Square

Let  $Q_1$  and  $Q_2$  be squares contained in a unit square, with  $Q_i$  having side-length  $a_i$ . If  $Q_1$  and  $Q_2$  have disjoint interiors then  $a_1 + a_2 \leq 1$ .

*Proof.* We may assume that  $Q_1$  and  $Q_2$  are as in Figure 39, i.e.,  $Q_1$  has a vertex on the bottom side and a vertex on the left, and  $Q_2$  has a vertex on the top side and on the right. Indeed, either  $Q_1$  can be slid to the left until it touches the left side, and  $Q_2$  to the right, until it touches the right side, or the other way round. Also, we can do the same for the top and bottom sides.

Let  $ABCD$  be a unit square, touching the axes as in Figure 40. We may and shall assume that  $\alpha$ , the angle  $\angle DAE$ , is at most  $\pi/4$ . Let  $O'$  be the projection of  $C$  onto the line  $OB$ . The triangle  $BO'C$  is congruent to  $BOA$ , so  $C$  is above the bisector  $\ell$  of the  $\pi/2$  angle at  $O$ . Hence,  $\ell$  meets the side  $CD$  in a point  $F$ . Let  $E$  be the intersection of the line  $CD$  with the  $x$  axis, as in Figure 40.

We claim that  $OF$  is at least  $\sqrt{2}$ . This implies the assertion since then the diagonal has two disjoint segments, one of length at least  $a_1\sqrt{2}$ , and the other of length at least  $a_2\sqrt{2}$ , so that  $a_1\sqrt{2} + a_2\sqrt{2} \leq \sqrt{2}$ , i.e.,  $a_1 + a_2 \leq 1$ .

Now, to see the claim, let  $\alpha = \angle DAE$ , so that  $\angle AED = \pi/2 - \alpha$ , and  $\angle EFO = \pi - \pi/4 - (\pi/2 - \alpha) = \pi/4 + \alpha$ . From the triangles  $AOB$  and  $ADE$  we see that  $OA = \sin \alpha$  and  $AE = 1/\cos \alpha$ . Appealing to the sine theorem, from the triangle  $OEF$  we see that

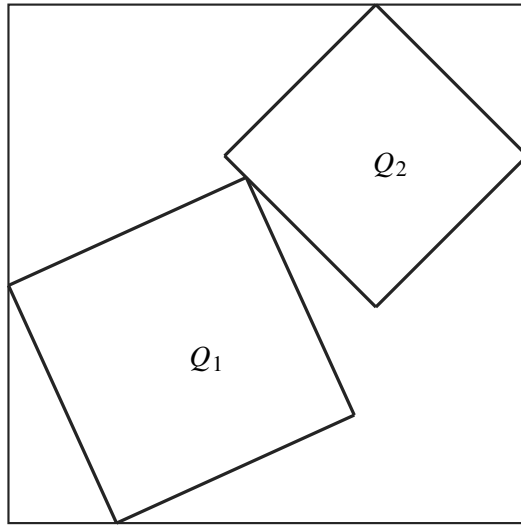


Fig. 39. Two squares,  $Q_1$  and  $Q_2$ , in a unit square.

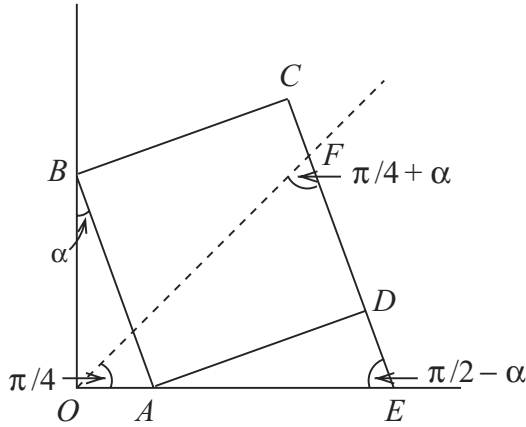


Fig. 40. A unit square  $ABCD$  in the first quadrant, with  $A$  on the  $x$ -axis,  $B$  on the  $y$ -axis; the angle  $DAE$  is  $\alpha$ , the bisector  $\ell$  of the  $\pi/2$  angle at  $O$  meets the line  $CD$  in  $F$ , and the line  $CD$  meets the  $x$ -axis in  $E$ .

$$OF = \frac{\sin(\pi/2 - \alpha)}{\sin(\pi/4 + \alpha)} \left( \sin \alpha + \frac{1}{\cos \alpha} \right) = \sqrt{2} \frac{\sin \alpha \cos \alpha + 1}{\cos \alpha + \sin \alpha} \geq \sqrt{2},$$

since

$$\sin \alpha \cos \alpha + 1 - \cos \alpha - \sin \alpha = (1 - \cos \alpha)(1 - \sin \alpha) \geq 0.$$

This proves the claim.  $\square$

**Notes.** This is another of the questions Paul Erdős used to pose to ‘epsilons’ (in particular, to me in 1958). In fact, this question is the first (and essentially trivial) case of a problem Erdős posed in 1932. For  $r \geq 1$ , denote by  $f(r)$  the supremum of the sum of the side-lengths of  $r$  non-overlapping squares inscribed in a unit square. By compactness,  $f(r)$  is attained by some packing if we allow zero side-lengths, or at most  $r$  squares. The full (and pretty hopeless) problem is to evaluate  $f(r)$ . If  $a_1, \dots, a_r$  are the side-lengths of  $r$  non-overlapping squares in a unit square then

$$\sum_{i=1}^r a_i^2 \leq 1,$$

so, by the Cauchy–Schwarz inequality,

$$\sum_{i=1}^r a_i \leq \sqrt{r}.$$



In particular,  $f(r) \leq \sqrt{r}$ . The partition of the unit square into  $k^2$  congruent squares (i.e., the  $k \times k$  grid) shows that  $f(k^2) \geq k$ ; hence,  $f(k^2) = k$ . Also, since  $f(r)$  is monotone increasing,

$$\lfloor \sqrt{r} \rfloor \leq f(r) \leq \sqrt{r}$$

for every  $r \geq 1$ .

Not surprisingly, Erdős did not ask for the exact determination of the function  $f(r)$ : that would have been boring and unreasonable. What he did in 1932 was much more interesting: he conjectured that

$$f(k^2) = f(k^2 + 1)$$

for every natural number  $k$ . The question we have just answered is the trivial  $k = 1$  case of Erdős' conjecture. Although this conjecture does *not* seem hopeless, in addition to  $k = 1$ , only the case  $k = 2$  has been settled so far, by D.J. Newman.

In 1995, Erdős and Soifer (and, independently, D.G. Rogers and S. Suen) proved that the function  $f(r)$  is strictly increasing at every other place: if  $r$  is not a perfect square then  $f(r + 1) > f(r)$ . Ten years later this result was rediscovered by Campbell and Staton. Recently, Rogers proved the following *Bordering Lemma* about the behaviour of  $f(r)$ .

Let  $r$  and  $s$  be positive integers. Then

$$(s + 1)f(r + 2s + 1) \geq sf(r) + 2s + 1.$$

Rogers used this lemma to show that the values for which Erdős' conjecture holds form an initial segment of the natural numbers: if  $1 \leq k \leq \ell$  and  $f(\ell^2 + 1) = f(\ell^2)$  then  $f(k^2 + 1) = f(k^2)$ .

Needless to say, the problem discussed here can be generalized to higher dimensions and other shapes, like simplices; in particular, Conway and Soifer have posed its analogue for triangles.

C. Campbell and W. Staton, A square-packing problem of Erdős, *Amer. Math. Monthly* **112** (2005), 165–167.

J. H. Conway and A. Soifer, Covering a triangle with triangles, *Amer. Math. Monthly* **112** (2005), 78.

P. Erdős and A. Soifer, Squares in a square, *Geombinatorics* **4** (1995), 110–114.

D.G. Rogers, A conjecture of Erdős from 1932: packing squares in a square, manuscript

### 33. Lines Through Points: the Sylvester–Gallai Theorem

*Given a finite set of points in a Euclidean space, not all on a line, there is a line that contains exactly two of the points.*

This well known, beautiful, and very influential result has a colourful history. The problem whether there is a line with precisely two points was posed by J.J. Sylvester as *Mathematical Question 11851* in the *Educational Times*, **59** (1893), p. 98. Forty years later, the question was rediscovered by Paul Erdős, and answered in the affirmative by Erdős' close friend, Tibor Grünwald (later Gallai). In the 1950s Erdős frequently posed this question to *epsilons*; not surprisingly, this is how I first heard of the problem as well.

When in 1943 Erdős published this problem in *The American Mathematical Monthly*, several other beautiful solutions were found. To demonstrate the great variety of approaches to the Sylvester–Gallai theorem, as the result is usually known now, we shall present four proofs.

In all the proofs, we may assume that the points are in a plane, since we may project them onto a plane so that they remain non-collinear.

*First Proof.* This is Gallai's original proof, as reproduced by de Bruijn and Erdős in 1948.

Suppose that there is a finite set of points not all on a line such that any line through two of the points also goes through a third. We shall show that this assumption leads to a contradiction.

To start Gallai's proof, project one of the points,  $a_1$ , say, to infinity, and connect it with the other points. Thus we get a set of parallel lines each containing two or more of the points  $a_i$  (in the finite part of the plane). Consider the system of lines connecting any two of these points, and assume that the line  $\ell$  forms the smallest angle with the parallel lines. By assumption,  $\ell$  goes through at least three of the points; say,  $a_i$ ,  $a_j$  and  $a_h$  are on  $\ell$  in this order (see Figure 41). But the line connecting  $a_j$  with  $a_1$  (at infinity) contains at least another (finite) point,  $a_r$ , say, and either the line through  $a_r$  and  $a_i$ , or the line through  $a_r$  and  $a_h$  forms a smaller angle with the parallel lines than  $\ell$ . This contradiction completes the proof.  $\square$

Although Gallai's proof above is ingenious and elegant, in simplicity it cannot compete with the following proof given by Kelly in 1947, reproduced by Coxeter in his paper on this problem.

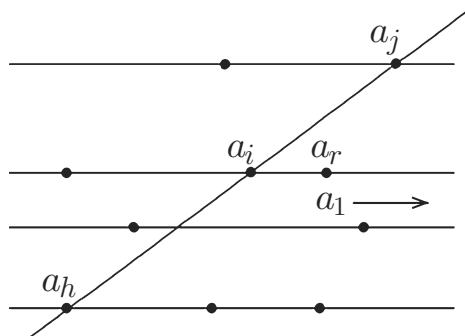


Fig. 41. The lines through the point  $a_1$  at infinity, and the line  $\ell$  through  $a_h$ ,  $a_i$  and  $a_j$  forming a minimal angle with these lines.

*Second Proof.* Let  $\mathcal{P}$  be the set of our points and  $\mathcal{L}$  the set of lines determined by the pairs of our points. Let  $\ell \in \mathcal{L}$  be a line and  $x \in \mathcal{P}$  a point *not* on  $\ell$  such that their distance is minimal (among the strictly positive distances).

To complete our solution, we shall show that  $\ell$  contains only two points of  $\mathcal{P}$ . Let  $f$  be the foot of the perpendicular dropped from  $x$  onto  $\ell$ , and let  $a$  and  $b$  be two points of set  $\mathcal{P}$  on  $\ell$ . Can  $a$  and  $b$  be on the same side of  $f$ ? Clearly not, since if  $a, b$  and  $f$  are in this order on  $\ell$  (with  $b = f$  allowed), then  $b$  would be closer to the line of  $xa$  than  $x$  is from  $\ell$ . Consequently, each of the two *closed* half-lines of  $\ell$  determined by  $f$  contains at most one point of  $\mathcal{P}$ . Hence,  $\ell$  contains exactly two points of  $\mathcal{P}$ .  $\square$

A slightly different description of the end of Kelly's proof goes as follows. If  $\ell$  contained three points (and, perhaps, some others), say  $a, b$  and  $c$ , in this order, then the sum of the angles  $abx$  and  $cbx$  would be  $\pi$ , so we could assume that the angle  $abx$  is at least  $\pi/2$ . In particular, in the triangle  $abx$  the angle at  $b$  would be larger than the one at  $x$ , so  $b$  would be closer to the line through  $a$  and  $x$  than  $x$  is from  $\ell$ , the line through  $a$  and  $b$ . The result follows.

The third proof we shall present is essentially due to Robert Steinberg, who transformed Gallai's projective proof into an affine one. Coxeter further transformed this proof into one that uses neither parallelism nor distance, only the single primitive entity *point* and the single primitive relation *between*. Thus this proof satisfies more stringent conditions than the previous two.

*Third Proof.* Let  $n$  points in the plane have the property that the line joining any two of them passes through a third point of the set. We shall show that our points are collinear.

Suppose that this is not the case, i.e., some three of our points form a triangle  $ABC$ . Our aim is to arrive at a contradiction.

Let  $P'$  be a point of the line  $BC$  such that the only point of our set on the line  $AP'$  is  $A$ . The lines through pairs of points meet this line  $AP'$  in  $A$ ,  $P'$ , and possibly some (finitely many) other points. Let  $P$  be the *first* of these points on the segment  $AP'$  after  $A$ , when going from  $A$  towards  $P'$ . Thus if the open segment  $AP'$  contains no intersection point then  $P = P'$ . By definition, the open segment  $AP$  contains no intersection point: our aim is to prove the contradiction that it does.

As every line through two of the points contains at least three points, the point  $P$  lies on a line containing at least three of our points, say  $Q$ ,  $R$  and  $S$ . The notation can be chosen so that the segment  $PR$  contains  $Q$  but not  $S$ . (Note that the order of the points  $P$ ,  $Q$ ,  $R$  and  $S$  is either this or  $S$ ,  $P$ ,  $Q$ ,  $R$ .) The line through  $A$  and  $R$  contains at least one more point,  $O$ , say. Where is this point  $O$ ?

If it is between  $A$  and  $R$ , as in Figure 42, then the line  $SO$  must intersect the segment  $AP$ . Similarly, if  $O$  lies outside the segment  $AR$ , as in Figure 43, the line  $QO$  must intersect the segment  $AP$ . In either case, the open segment  $AP$  contains an intersection point, contradicting our assumption.  $\square$

The only result we used in the argument above is Pasch's theorem: *given a triangle  $XYZ$ , a point  $U$  on the half-line  $YZ$  outside the closed segment  $YZ$ , and a point  $V$  in the open segment  $XY$ , the line  $UV$  meets the open segment  $XZ$ .*

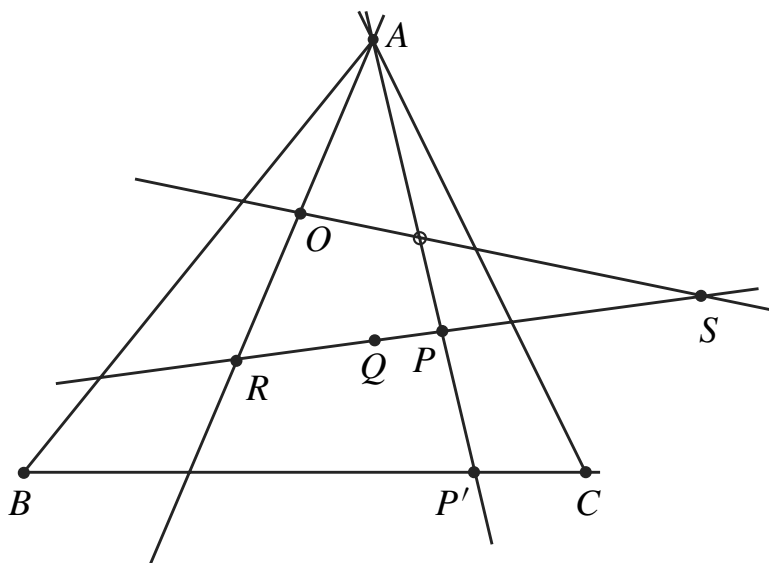


Fig. 42. If the open segment  $AR$  contains a point  $O$ , then the line  $OS$  meets the open segment  $AP$ .

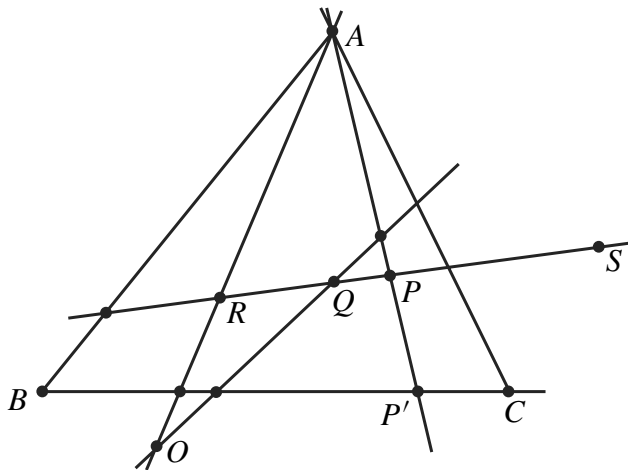


Fig. 43. If the half-line  $AR$  contains a point  $O$  outside the closed segment  $AR$ , then the line  $OQ$  meets the open segment  $AP$ .

The fourth and final proof we shall give was found by the topologist Norman Steenrod, who fondly remembered it when I reminded him of it in 1970.

*Fourth Proof.* Suppose there is no line through precisely two points. Let us dualize our points and the lines determined by pairs of our points: first project the points into pairs of antipodal points on a sphere and the lines into great circles; then replace every great circle by the pair of antipodal points determining it (so that the equator is replaced by the North and South poles), and every pair of antipodal points by the great circle they determine. Consider the obtained set of points and lines as a spherical graph: the vertices are the points and the edges are those arcs of our great that join two points and contain no other points. By assumption, every vertex of our graph has degree at least six, since every point is on at least three great circles, and a great circle through a point gives at least two edges. However, this contradicts the immediate consequence of Euler's theorem that a planar graph on  $n$  vertices has at most  $3n - 6$  edges, and so has average degree less than six.  $\square$

**Notes.** In the original *Monthly* problem in 1943, Paul Erdős also asked for a proof of the following easy consequence of the Sylvester–Gallai theorem.

*Let there be  $n$  points in the plane, not all on a line. Connect any two of the points. Then the number of lines in this system is at least  $n$ . Equality occurs only if  $n - 1$  of the points are on a line.*

To see this, apply induction on  $n$ . Assume that any  $n - 1 \geq 3$  points, not all on a line, determine at least  $n$  lines, unless  $n - 2$  of them are collinear, in which case they determine  $n - 1$  lines. Given  $n$  points,  $a_1, a_2, \dots, a_n$ , not all on a line, by the Sylvester–Gallai theorem there are two points,  $a_1$  and  $a_2$ , say, such that the line through them contains no other point of our set. If the points  $a_2, \dots, a_n$  are all on a line  $\ell$  then we have the  $n$  lines  $a_1a_2, a_1a_3, \dots, a_1a_n$  and  $\ell$ , as claimed. Otherwise, by the induction hypothesis, the points  $a_2, \dots, a_n$  determine at least  $n$  lines, unless  $n - 2$  of them are collinear, and  $a_1a_2$  is not one of these lines. Finally, if  $n - 2$  of the points  $a_2, \dots, a_n$  are collinear then we have  $2(n - 2) + 1 = 2n - 3 > n$  lines.

A little later de Bruijn and Erdős extended this result to the *de Bruijn–Erdős theorem* in Problem 49.

Much work has been done on strengthening the Sylvester–Gallai theorem in other directions as well. Given a finite set of points in the plane, call a line *ordinary* if it is incident with precisely two of the points. Let  $m(n)$  be the minimal number of ordinary lines in an arrangement of  $n$  points not all on a line. The Sylvester–Gallai theorem tells us that  $m(n) \geq 1$ . But how large is  $m(n)$ ? In 1951, Dirac proved that  $m(n) \geq 3$  whenever  $n \geq 3$ , and Motzkin showed that  $m(n) \geq \sqrt{2n - 2}$ . In 1958, these results were improved considerably by Kelly and Moser, when they proved that  $m(n) \geq 3n/7$ , with equality for  $n = 7$ , as shown by the Fano plane's usual drawing in the plane in Figure 44. Dirac conjectured that when  $n \neq 7$ ,  $m(n) \geq n/2$ . In 1968, Crowe and McKee disproved this general conjecture by showing that  $m(13) \leq 6$ , and conjectured that this is the only other exceptional case, i.e.,  $m \geq n/2$  when  $n \neq 7, 13$ . Furthermore, they made use of examples due to Böröczky to show that for  $n$  even this would be best possible since  $m(n) \leq k$  when  $n = 2k$  and  $m(n) \leq 3k$  when  $n = 4k + 1$  or  $4k + 3$ .

In 1981, Hansen gave a long and very complicated argument purporting to prove Dirac's modified conjecture. Although Hansen's proof was received with some

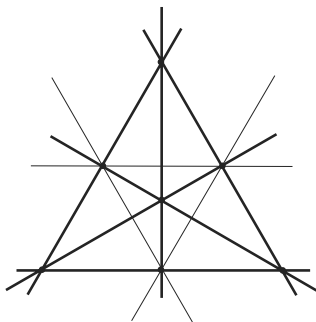


Fig. 44. An arrangement of seven points with three ordinary lines.

scepticism, it took twelve years for a fault to be discovered, when Csima and Sawyer worked through the proof. They also improved the Kelly–Moser bound to  $6n/13$ ,  $n \neq 7$ . A little later, Csima and Sawyer simplified their own proof. Nevertheless, the conjecture that  $m(n) \geq n/2$  if  $n \neq 7, 13$  is still open.

For an account of the early results, see the 1990 review article by Borwein and Moser; see also Chapter 8 of the book by Aigner and Ziegler.

M. Aigner and G. Ziegler, *Proofs from The Book*, with illustrations by K.H. Hofmann, Third edition, Springer-Verlag, (2004), viii + 239 pp.

P.B. Borwein and W.O.J. Moser, A survey of Sylvester's problem and its generalizations, *Aequationes Math.* **40** (1990), 111–135.

N.G. de Bruijn and P. Erdős, On a combinatorial problem, *Indagationes Math.* **10** (1948), 421–423.

H.S.M. Coxeter, A problem of collinear points, *Amer. Math. Monthly*, **55** (1948), 26–28.

D.W. Crowe and T.A. McKee, Sylvester's problem on collinear points, *Math. Mag.* **41** (1968), 30–34.

J. Csima and E.T. Sawyer, There exist  $6n/13$  ordinary points, *Discrete Comput. Geom.* **9** (1993), 187–202.

J. Csima and E.T. Sawyer, The  $6n/13$  theorem revisited, in *Graph theory, Combinatorics, and Algorithms*, Vol. 1 (Kalamazoo, MI, 1992), Wiley, (1995), pp. 235–249.

G.A. Dirac, Collinearity properties of sets of points, *Quart. J. Math., Oxford Ser. (2)* **2** (1951), 221–227.

P. Erdős, Three point collinearity, Problem 4065, *Amer. Math. Monthly* **50**, (1943), 65. Solutions in volume **51** (1944), 169–171.

S. Hansen, *Contributions to the Sylvester–Gallai Theory*, Ph.D. Dissertation, University of Copenhagen, 1981, 194 pp.

L.M. Kelly and W.O.J. Moser, On the number of ordinary lines determined by  $n$  points, *Canad. J. Math.* **1** (1958), 210–219.

T. Motzkin, The lines and planes connecting the points of a finite set, *Trans. Amer. Math. Soc.* **70** (1951), 451–464.

### 34. The Spread of Infection on a Square Grid

*The minimal number of infected sites needed to infect eventually all  $n^2$  sites of the  $n \times n$  grid is  $n$ .*

*Proof.* That  $n$  infected sites suffice is shown by the diagonal: if all sites on the diagonal are infected then after the first step all sites within one of the diagonal are infected, after the second, all within distance two, and so on. The real content of this assertion is, of course, that  $n$  infected sites are needed to infect all the squares.

Let us replace the grid by a ‘chess board’ (without its colouring), so that the sites become the (unit) squares of the board; also, colour an infected square black and a healthy one white. In particular, the number of infected sites is the area of the black domain, as in Figure 45.

Note when a healthy square (with at least two infected neighbours) becomes infected then the *perimeter length* of the black domain does not increase. Since at the end the entire  $n \times n$  board is black, so has perimeter length  $4n$ , the perimeter length of the black area we start with is at least  $4n$ , so it consists of at least  $n$  squares. Thus we do need at least  $n$  infected squares for the process to end with all squares infected.  $\square$

**Notes.** The argument above shows considerably more, namely that a  $k \times \ell$  rectangular grid needs at least  $\lceil (k + \ell)/2 \rceil$  infected sites to infect every site. As shown in Figure 46, it is obvious that a suitable arrangement of this many sites will ensure that every site becomes infected. In fact, assuming that  $k < \ell$ , all we need is to use  $k$  infected sites to spread the disease on the entire  $k \times k$  square, and add to it  $\lceil (\ell - k)/2 \rceil$  infected sites so that every second column has an infected site, and so does the last column. This beautiful argument is probably folklore, and seems to have been first published (at least in the West) by Balogh and Pete.

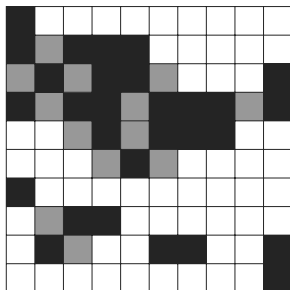


Fig. 45. One step in the spread of a disease; the black cells have already been infected, the lightly shaded ones are the newly infected cells.



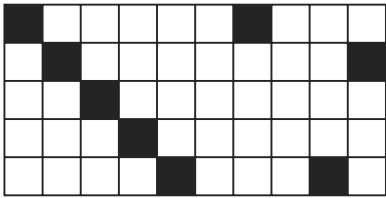


Fig. 46. Starting with eight infected sites, as above, all sites of a  $5 \times 10$  board will be infected; note that  $8 = \lceil (5 + 10)/2 \rceil$ .

J. Balogh and G. Pete, Random disease on the square grid, in *Proceedings of the Eighth International Conference ‘Random Structures and Algorithms’* (Poznań, 1997), *Random Structures Algorithms* **13** (1998), 409–422.

### 35. The Spread of Infection in a $d$ -dimensional Box

The minimal number of infected sites needed to infect eventually all  $n^d$  sites in the  $d$ -dimensional  $n \times n \times \cdots \times n$  box is  $\lceil d(n-1)/2 \rceil + 1$ .

*Proof.* The assertion is trivial for  $d = 1$  (and in the solution to Problem 34 we showed it for  $d = 2$ ), so in what follows we may assume that  $d \geq 2$ .

We shall do a little more than strictly required by the result: we shall determine the minimum for any  $d$ -dimensional box

$$B = \prod_{i=1}^d [a_i, b_i) = \{(x_i)_1^d \in \mathbb{Z}^d : a_i \leq x_i < b_i, i = 1, \dots, d\},$$

where  $(a_i)_1^d, (b_i)_1^d \in \mathbb{Z}^d$ , with  $a_i < b_i$  for every  $i$ . In what follows, we shall always assume that our boxes are non-empty, although occasionally we shall emphasize this.

Somewhat unconventionally, we shall use a linear function to measure the size of a box: if  $B$  is  $\prod_{i=1}^d [a_i, b_i)$ , as above, then we set

$$\ell(B) = \sum_{i=1}^d (b_i - a_i) - (d - 2).$$

Setting  $n_i = b_i - a_i$ , clearly  $B$  is a  $n_1 \times \cdots \times n_d$  box, with  $\prod_{i=1}^d n_i$  elements (sites, lattice points). In particular, if  $B$  is an  $n \times \cdots \times n$  box, as in our problem, then  $\ell(B) = d(n-1) + 2$ ; thus, the measure of a box containing a single site is 2, which may be a little strange. Actually, this ‘linear measure’ is trivially subadditive in the sense that if a box  $B$  is the union of some boxes  $B_1, \dots, B_r$  then  $\ell(B) \leq \sum_{i=1}^r \ell(B_i)$ . Also, this inequality is strict if  $r \geq 2$ .

As we shall see, in this notation, for a general box  $B$  the answer to the question is  $\lceil \ell(B)/2 \rceil$ : this is the minimal number of sites in  $B$  that can infect the entire box  $B$ . The main task is to show that we need *at least this many* sites: this is what we shall do first; it will be rather easy to show that this many sites do suffice.

We shall prove that at least  $\lceil \ell(B)/2 \rceil$  sites are needed by describing the infection process in a fairly systematic way; this description is based on the structure of a set from which the infection does not spread to a larger set.

Let us say that a set  $S$  *infects*  $S'$ , in notation,  $S \rightarrow S'$ , if all the sites in  $S'$  can be infected from  $S$ . Also,  $S$  *spans*  $S'$  if  $S \subset S'$  and  $S \rightarrow S'$ . Our aim then is to prove that if  $S$  spans a box  $B$  then  $S$  has at least  $\ell(B)/2$  elements.

Call a set  $S$  *closed* if  $S'$  is the only set spanned by  $S$ . Every set  $S$  is contained in a minimal closed set  $\bar{S}$  containing it: this is the intersection of all closed sets containing  $S$ , and also the maximal set infected by  $S$ , i.e., the union of all sets that

can be infected if we start from  $S$ . We call  $\bar{S}$  the *closure* of  $S$ . Clearly,  $S$  is closed if and only if  $\bar{S} = S$ . Note that every box is closed, so we have to show that if  $\bar{S} = B$  then  $S$  has at least  $\ell(B)/2$  elements.

Let us use the  $\ell_1$ -norm to measure the *distance* in  $\mathbb{Z}^d$ : given  $x = (x_i)_1^d, y = (y_i)_1^d \in \mathbb{Z}$ , set

$$d(x, y) = \|x - y\|_1 = \sum_{i=1}^d |x_i - y_i|.$$

Also, the distance between two non-empty sets  $S, T \subset \mathbb{Z}^d$  is

$$d(S, T) = \min\{d(s, t) : s \in S, t \in T\}.$$

Clearly, the measure  $\ell(B)$  of a box  $B$  is rather close to its *diameter*  $\text{diam}(B)$  defined in the usual way:

$$\text{diam}(B) = \max\{d(x, y) : x, y \in B\};$$

indeed,  $\ell(B) = \text{diam}(B) + 2$ .

Note that for any two boxes  $B$  and  $B'$  there is a unique minimal box containing both, the *join*  $B \vee B'$  of the boxes: if  $B = \prod [a_i, b_i)$  and  $B' = \prod [c_i, d_i)$  then

$$B \vee B' = \prod_{i=1}^d [e_i, f_i)$$

where  $e_i = \min\{a_i, c_i\}$  and  $f_i = \max\{b_i, d_i\}$ . Note also, that if  $d(B, B') \geq 3$  then the set  $B \cup B'$  is closed; on the other hand, if  $d(B, B') \leq 2$  then

$$\overline{B \cup B'} = B \vee B'. \quad (3)$$

Although relation (3) is essentially trivial, we shall justify it in great detail. First, note that it suffices to check it in the case when  $B'$  consists of a single site  $z$  at distance 1 from  $B$ . Indeed, having proved this simple case, the general case follows by taking the closures with more and more points added to the set: if  $d(B, B') \leq 2$  then there are points  $w_0, w_1, \dots, w_N$  such that  $d(B, w_0) \leq 1$ , and, for every  $i, 1 \leq i \leq N$ , the site  $w_i$  is at distance at most 1 from the closure of  $B \cup \{w_1, \dots, w_{i-1}\}$ , which is itself a box.

Now, to check (3) for  $B' = \{z\}$  with  $d(B, z) = 1$ , note that we may assume that  $z$  is just above the top face of  $B$ . Then it is easily seen that this face and  $z$  together will infect the entire layer on top of the face. Formally, set  $F = \prod_{i=1}^{d-1} [a_i, b_i)$ , so that  $F \times \{b_d - 1\}$  is the top face of  $B$ , and let  $z = (z_i)_1^d$  be such that  $z_d = b_d$  and  $a_i \leq z_i < b_i$  for  $1 \leq i < d$ . Then on  $F \times \{b_d\}$  the set of sites that can be infected by  $B \cup \{z\}$  includes  $z \in F \times \{b_d\}$  and is a connected set, so it contains (actually, is) the entire layer  $F \times \{b_d\}$  just above the top face of  $B$ .

What we have said so far gives the following beautiful characterization of the closed finite subsets of  $\mathbb{Z}^d$ . Given a (finite) set  $S$ , write it as the union of as few boxes as possible; say, let  $S = \bigcup_{i=1}^r B_k$ , where each  $B_k$  is a box. If  $d(B_i, B_j) \geq 3$  for all  $1 \leq i < j \leq r$  then  $S$  is closed:  $S$  cannot infect any new site since no site can be infected ‘jointly’ by two boxes, and each box is closed. Now, suppose that  $S$  is closed but  $d(B_i, B_j) \leq 2$  for some indices  $1 \leq i < j \leq r$ . Then  $\overline{B_i \cup B_j} = B_i \vee B_j$  is a box  $B_{ij}$ , and so  $S = B_{ij} \cup \bigcup_{k \neq i, j} B_k$ , contradicting the assumption that  $S$  is not a union of  $r - 1$  boxes. Thus,

*a set is closed if it is the union of a set of boxes, with any two at distance at least 3.*

As the final preparation for our proof, let us note an inequality concerning the linear measure of the join of two boxes. If for a box  $B$  and a site  $z$  we have  $d(B, z) = 1$  then

$$\ell(B \vee \{z\}) = \ell(B) + 1.$$

From this it follows that if  $B$  and  $B'$  are non-empty boxes then

$$\ell(B \vee B') \leq \ell(B) + \ell(B') + d(B, B') - 2.$$

In particular, we have the following companion of relation (3): if the (non-empty) boxes  $B$  and  $B'$  are within distance 2 then

$$\ell(\overline{B \cup B'}) = \ell(B \vee B') \leq \ell(B) + \ell(B'). \quad (4)$$

Although, strictly speaking, we have not even started our solution of the problem, after all this preparation our task will be very easy. In fact, we shall prove a little more. Define the *linear measure*  $\ell(S)$  of a finite set  $S \subset \mathbb{Z}^d$  as

$$\ell(S) = \min \left\{ \sum_{i=1}^r \ell(B_i) : S \text{ is the union of the boxes } B_1, \dots, B_r \right\}.$$

Since  $S$  is the union of single-site boxes, each of measure 2,  $\ell(S)$  is at most twice the number of sites in  $S$ . What we shall show is that if  $S$  can infect a box  $B$ , i.e.,  $S \rightarrow B$ , then

$$\ell(B) \leq \ell(S). \quad (5)$$

Since  $\ell(S)$  is at most twice the number  $|S|$  of sites in  $S$ , inequality (5) is indeed more than the inequality  $2|S| \geq \ell(B)$  we have set out to prove.

Now, to prove (5), given a box  $B$ , let  $S \subset B$  be such that

- (i)  $S \rightarrow B$ ,
- (ii)  $\ell(S)$  is minimal,
- (iii)  $S = \bigcup_{k=1}^r B_k$  with  $\ell(S) = \sum_{k=1}^r \ell(B_k)$  and  $r$  minimal.

If  $r = 1$  then  $B = B_1$  and we are done:  $\ell(S) = \ell(B)$ ; otherwise, as  $S$  is not closed,  $d(B_i, B_j) \leq 2$  for some two boxes with  $1 \leq i < j \leq r$ . Imitating the closure-argument above, write  $B_{ij}$  for the box  $\overline{B_i \cup B_j} = B_i \vee B_j$ , set  $S' = B_{ij} \cup \bigcup_{k \neq i, j} B_k$ , and note that we have  $S \rightarrow S' \rightarrow B$  and, by (4),

$$\ell(S) \geq \ell(B_{ij}) + \sum_{k \neq i, j} \ell(B_k) \geq \ell(S').$$

Since in the representation above  $S'$  is the union of only  $r - 1$  boxes, the existence of the triple  $(S', \ell(S), r - 1)$  contradicts our choice of  $(S, \ell(S), r)$ , and so our proof of (5) is complete. In particular, in order to infect an  $n_1 \times \cdots \times n_d$  box  $B$  we have to start with at least  $\lceil \ell(B)/2 \rceil = \left\lceil \left( \sum_{i=1}^d n_i - d + 2 \right) / 2 \right\rceil$  infected sites.

Now, to see that this many infected sites suffice, let us apply induction on  $d$  to the assertion that a set  $S$  of  $\lceil \sum_{i=1}^d (n_i - 1)/2 \rceil + 1$  sites can infect the entire box

$$B = \prod_{i=1}^d [n_i] = \{(x_i)_1^d \in \mathbb{Z}^d : 1 \leq x_i \leq n_i\},$$

with  $\mathbf{1} = (1, 1, \dots, 1)$  as one of the sites in  $B$ . Needless to say, we may always assume that  $n_i \geq 2$  for every  $i$ . From the solution of Problem 34 we know that this holds for  $d \leq 2$ , so we turn to the induction step. Suppose that  $d \geq 3$ ,  $n_i \geq 2$  for every  $i$ , and the assertion holds for smaller values of  $d$ . Since  $d \geq 3$ , we may and shall assume that  $n_1 + n_2$  is even. Let  $S_1$  be a set of  $(n_1 + n_2)/2$  sites that infect the entire board  $[n_1] \times [n_2]$  with  $(1, 1) \in S_1$ . Also, let  $S_2$  be a set of  $\sum_{i=3}^d (n_i - 1)/2 + 1$  sites infecting  $[n_3] \times \cdots \times [n_d]$ , with  $(1, \dots, 1) \in S_2$ . Then the union of the sets

$$\{(z_1, z_2, 1, \dots, 1) : (z_1, z_2) \in S_1\}$$

and

$$\{(1, 1, z_3, \dots, z_d) : (z_3, \dots, z_d) \in S_2 \setminus \{\mathbf{1}\}\}$$

clearly infects  $B$ , so our proof is complete. □

### 36. Sums of Integers: an Easy Erdős Problem for Epsilons

Let  $1 \leq a_1 < a_2 < \dots < a_\ell \leq n$  be integers with  $\ell > (n+1)/2$ . Then  $a_i + a_j = a_k$  for some  $1 \leq i < j < k \leq \ell$ .

*Proof.* Set  $A = \{a_1, a_2, \dots, a_\ell\}$  and  $B = A - a_1 = \{a_2 - a_1, a_3 - a_1, \dots, a_\ell - a_1\}$ . Then  $A$  and  $B$  are subsets of the first  $n$  natural numbers; since  $|A| + |B| = \ell + (\ell - 1) > n$ , the intersection  $A \cap B$  is not empty, that is,  $a_j = a_k - a_1$  for some  $j$  and  $k$  with  $1 < j < k \leq \ell$ , proving the assertion.  $\square$

**Notes.** The condition  $\ell > (n+1)/2$  is best possible, since for  $\ell = \lfloor (n+1)/2 \rfloor$  the equation  $a_i + a_j = a_k$  is not solvable in the set of  $\ell$  numbers from  $a_1 = n - \ell + 1 = \lceil (n+1)/2 \rceil$  to  $a_\ell = n$ .



### 37. Normal Numbers: the Champernowne Number

*The real number*

$$\gamma = 0.1\,2\,3\,\dots\,9\,10\,11\,\dots\,99\,100\,101\,\dots,$$

whose significant digits are formed by the concatenation of all the natural numbers, is normal in base 10. This number  $\gamma$  is called the Champernowne number.

*Proof.* As in the Problem, given a word  $w$ , write  $f_n(w; \gamma)$  for the number of times the word of the first  $n$  significant digits of  $\gamma$  contain  $w$ . Note that a natural number  $\ell$  has  $\lceil \log_{10}(\ell + 1) \rceil$  digits in base 10, so the total number of digits in the first  $m$  natural numbers is

$$n(m) = \sum_{\ell=1}^m \lceil \log_{10}(\ell + 1) \rceil \sim m \log_{10} m.$$

Hence, it suffices to show that, for a fixed value of  $k$ , as  $m \rightarrow \infty$ , the concatenation of the first  $m$  natural numbers contains every  $k$ -letter word  $w = w_1 \dots w_k$  the right number, about  $m \log_{10} m / 10^k$ , of times. Even more, we may assume that  $m$  ends in ‘many’ 0s:  $m = a10^b$ , where  $b/10 \leq a \leq b$ , say. (Indeed, if  $m'$  is obtained from  $m$  by changing the last  $\log_{10} m - \log \log m$  digits to 0 then  $m' \sim m$  as  $m \rightarrow \infty$ , and  $m'$  is of the form  $m' = a10^b$  with  $b/10 \leq a \leq b$ . In fact, for the argument below, it would suffice to have  $m$  in the form  $m = a10^b$  with  $\log a = o(b)$ .) Note that for such an  $m$  we have  $\log_{10} m \sim b$ , so  $n(m) \sim a b 10^b$ .

Having made this choice of  $n = n(m)$ , we are almost done. For  $1 \leq c \leq a$ , the total number of times the  $10^b$  numbers  $c 10^b, c 10^b + 1, \dots, (c + 1) 10^b - 1$  contain a  $k$ -term sequence  $w = w_1 \dots w_k$  among their last  $b$  digits is  $(b - k + 1)10^{b-k}$ . (Note that we allow any of the  $w_i$  to be 0; for example, 00010 is a permissible 5-term sequence.) Indeed, the last  $b$  digits of these numbers form all  $10^b$  sequences of length  $b$  with digits (letters) 0, 1, 2,  $\dots$ , 9. We have started with at least  $10^b$ , i.e., demanded that  $c \geq 1$ , to ensure that the sequences starting with 0s also occur (rather than shorter sequences).

*A fortiori*, the concatenation of the  $10^b$  numbers  $c 10^b + 1, c 10^b + 2, \dots, (c + 1) 10^b$  contains every  $k$ -letter word  $w = w_1 \dots w_k$  at least  $(b - k + 1)10^{b-k}$  times. Hence, for  $n = n(m)$  we have

$$f_{n(m)}(w; \gamma) \geq (a - 1)(b - k + 1) 10^{b-k} \sim m \log_{10} m / 10^k.$$

Since this holds for each of the  $10^k$  sequences  $w$  of length  $k$ , we have  $f_{n(m)}(w; \gamma) \sim m \log_{10} m / 10^k$ , completing the proof that  $\gamma$  is normal in base 10.  $\square$

**Notes.** The Champernowne number  $\gamma$  was constructed by the Cambridge mathematical economist David Champernowne when he was a student in Oxford, after a lecture of G.H. Hardy. As Champernowne related in later life, Hardy mentioned in a lecture that it was easy to see that almost every number is normal, but difficult to show that any *particular* number was normal. By the end of the lecture, Champernowne proved that his number was indeed normal.

It is hardly worth remarking that 10 can be replaced by any base  $b \geq 2$ ; the changes needed in the proof are utterly trivial.



### 38. Random Walks on Graphs

*Let  $s$  and  $t$  be vertices of a graph  $G$ . A random walk on  $G$  starts at  $s$  and stops at  $t$ . Then the expected number of times this walk traverses the edges of a cycle in one direction is equal to the expected number of traversals in the other direction.*

*Proof.* For each edge  $xy$ , let  $N(xy)$  be the expected number of traversals of the edge  $xy$  from  $x$  to  $y$ , and, for each vertex  $x$ , write  $N(x)$  for the expected number of visits to  $x$ . Clearly,  $N(xy) = N(x)/d(x)$ , where  $d(x)$  is the degree of  $x$ .

Hence, given a cycle  $C = x_1 \dots x_\ell$ , if we set  $x_{\ell+1} = x_1$  and  $x_0 = x_\ell$ , then we find that the two expectations are

$$\sum_{i=1}^{\ell} N(x_i x_{i+1}) = \sum_{i=1}^{\ell} N(x_i)/d(x_i) = \sum_{i=1}^{\ell} N(x_i x_{i-1}),$$

as claimed. □

J.L. Palacios, Going around randomly in circles, Problem 10881, *Amer. Math. Monthly* **108** (2001), p. 565; solutions in **110** (2003), p. 247.

### 39. Simple Tilings of Rectangles

*For  $n = 5$  and  $n \geq 7$  there are simple tilings of rectangles by rectangles; for the other values of  $n$  there are no simple tilings. For every  $n \geq 7$  there are at least  $2^{n/3}/4$  essentially different simple tilings with  $n$  rectangles.*

*Proof.* It is trivial that there is no simple tiling for  $n = 2, 3$  and  $4$ , and Figure 47 shows a simple tiling with five rectangles. It is somewhat cumbersome to show that for  $n = 6$  there is no simple tiling, and Figure 48 shows that for every  $n \geq 7$  there is a simple tiling with  $n$  rectangles.

Copies of the tiling shown in Figure 47 and its reflection can be glued together by uniting the two right-hand rectangles of one with the two left-hand rectangles of the next; in this way for every 0-1 sequence of length  $k$  we get a simple tiling with  $3k + 2$  rectangles, as shown in Figure 49. Since no more than four such sequences may give essentially the same sequence, we see that for  $n = 3k + 2$  we have at least  $2^{n/3}/8$  essentially different simple tilings with  $n$  rectangles. Starting with tilings

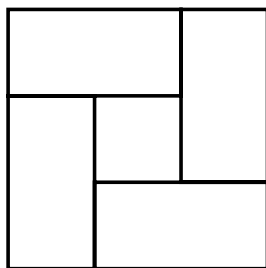


Fig. 47. A simple tiling with five rectangles.

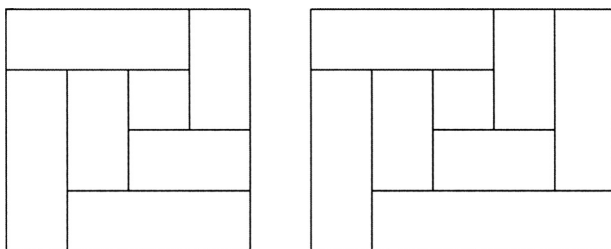


Fig. 48. Simple tilings with 7 and 8 rectangles; starting from the little square, we see a counter-clockwise spiral indicating how to turn a simple tiling with  $n \geq 7$  rectangles into one with  $n + 1$  rectangles.

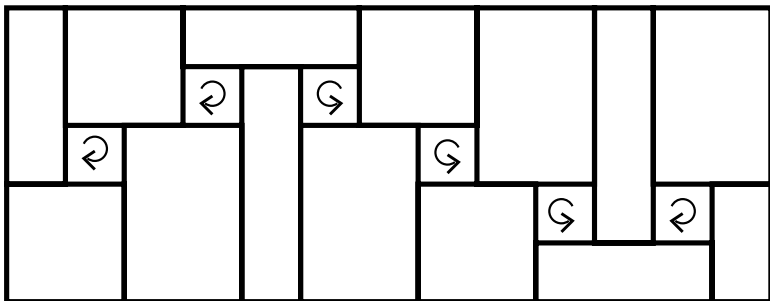


Fig. 49. A simple tiling with  $6 \cdot 3 + 2 = 20$  rectangles, corresponding to the sequence 110001.

shown in Figure 48 and their reflections for  $n = 7, 8$  and  $9$ , and continuing with 5-tilings, as before, we can do a little better: we find that for every  $n \geq 7$  there are at least  $2^{n/3}/4$  essentially different simple tilings with  $n$  rectangles, as claimed.  $\square$

**Notes.** The results are from Chung *et al.*, who proved also that if every rectangle has integer sides then, with the exception of the 5-tiling shown in Figure 47, in all other simple tilings the average area is greater than  $9/5$ . There is also a unique (up to translation, rotation and reflection) simple tiling of the plane in which the average area is  $9/5$ .

Chung, F.R.K., Gilbert, E.N., Graham, R.L., Shearer, J.B., and van Lint, J.H., Tiling rectangles with rectangles, *Mathematics Magazine* **55** (1982), 286–291.

## 40. *L*-tilings

*Cut out a square of a  $2^n$  by  $2^n$  chess board. Then the remaining  $2^{2n} - 1$  squares can be tiled with *L*-tiles, unions of three squares sharing a vertex.*

*Proof.* Let us apply induction on  $n$ . For  $n = 1$  the assertion is obvious, so let us turn to the induction step. Suppose that  $n \geq 1$  and the assertion holds for boards of size  $2^n$ . Let  $s$  be the square of the  $2^{n+1}$  by  $2^{n+1}$  board  $B$  to be cut out; our task is to find an '*L*-tiling' of  $B' = B - s$ . Divide the square  $B$  into four  $2^n$  by  $2^n$  boards,  $B_1, \dots, B_4$ , with  $B_4$  containing  $s$ , say. The three squares of  $B_1 \cup B_2 \cup B_3$  incident with the centre of  $B$  form an *L*-tile,  $L_1$ . With  $B'_i = B_i - L_1$ ,  $i = 1, 2, 3$ , and  $B'_4 = B_4 - s$ , each  $B'_i$  is a  $2^n$  by  $2^n$  board without one of the squares. By the induction hypothesis, each  $B'_i$  has an *L*-tiling: together with the single tile  $L_1$ , they form an *L*-tiling of  $B'$ , as in Figure 50.  $\square$

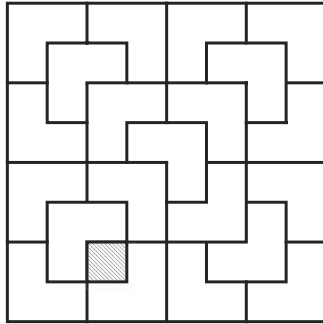


Fig. 50. An *L*-tiling of the  $8 \times 8$  board, from which the shaded square has been cut out. This tiling was constructed by the algorithm in the proof.

## 41. Antipodal Points and Maps: Borsuk's Theorem

The following four assertions are equivalent.

- (1) There is no continuous antipodal map  $S^n \rightarrow S^{n-1}$ .
- (2) Every continuous map  $S^n \rightarrow \mathbb{R}^n$  sends at least one pair of antipodal points into the same point.
- (3) Every continuous odd map  $S^n \rightarrow \mathbb{R}^n$  sends at least one point (and so at least one pair of antipodal points) into 0, the origin of  $\mathbb{R}^n$ .
- (4) For every family of  $n + 1$  closed sets covering  $S^n$ , one of the sets contains a pair of antipodal points.

*Proof.* In this proof we shall not be as economical as we could be: we start by proving that the first three assertions, concerning maps  $f : S^n \rightarrow \mathbb{R}^n$ , are best considered as a single result.

Note first that the implication (2)  $\Rightarrow$  (1) is trivial and, in fact, so are the implications (2)  $\Rightarrow$  (3) and (3)  $\Rightarrow$  (1). Indeed, the assertion that every continuous odd map  $f : S^n \rightarrow \mathbb{R}^n$  sends at least one point  $x$  into 0 means precisely that  $f$  cannot map  $S^n$  into a set not containing 0; in particular, it cannot map  $S^n$  into  $S^{n-1}$ . Also, suppose that a map  $f : S^n \rightarrow \mathbb{R}^n$  sends a pair of antipodal points  $x, -x$  into the same point. If  $f$  is odd then  $f(x) = f(-x) = -f(x)$ , so  $f(x) = f(-x) = 0$ .

To complete the proof of the equivalence of the first three assertions, we show that (1) implies (2). Suppose that  $f : S^n \rightarrow \mathbb{R}^n$  is a continuous map and  $f(-x) \neq f(x)$  for every  $x \in S^n$ . Then the map sending  $x$  into the point of  $S^{n-1}$  in the direction of  $f(x) - f(-x)$ , i.e., the map  $x \mapsto (f(x) - f(-x))/\|f(x) - f(-x)\|$ , is a continuous antipodal map.

It remains to show that the first three equivalent assertions are also equivalent to the fourth. To this end, we shall show that (2) implies (4) and (4) implies (1).

(2)  $\Rightarrow$  (4). Assume that (2) holds, and let  $F_1, \dots, F_{n+1}$  be closed sets covering  $S^n$ . Suppose that none of the sets  $F_1, \dots, F_n$  contains a pair of antipodal points. Setting

$$E_i = \{-x : x \in F_i\},$$

we find that none of the closed sets  $E_1, \dots, E_n$  contains a pair of antipodal points, i.e.,  $E_i \cap F_i = \emptyset$  for  $i = 1, \dots, n$ .

Let  $f_i(x) = d(x, E_i) - d(x, F_i)$ , where  $d(x, U)$  is the distance of a point  $x$  from a set  $U$ . Clearly, each function  $f_i$  is continuous, and  $f_i(x) > 0 > f_i(-x)$  for  $x \in F_i$  (and so  $-x \in E_i$ ). Therefore  $f = (f_1, \dots, f_n)$  is a continuous map  $S^n \rightarrow \mathbb{R}^n$ , so (2) implies that  $f(x_0) = f(-x_0)$  for some  $x_0 \in S^n$ . Then neither  $x_0$  nor  $-x_0$  is in  $\bigcup_{i=1}^n F_i$ . Consequently, both  $x_0$  and  $-x_0$  belong to  $F_{n+1}$ .

(4)  $\Rightarrow$  (1). Assume (4), and let  $f : S^n \rightarrow S^{n-1}$  be continuous. Let  $S^{n-1} \subset \bigcup_{i=1}^{n+1} E_i$ , with each  $E_i$  closed and diameter less than 2. It is easily seen that there

are such sets  $E_i$ . (For example, we may take for  $E_i$  the projection of the  $i$ th face of a regular simplex inscribed in  $S^{n-1}$  from the centre. In fact, we can do considerably better than this: writing  $B_1, \dots, B_n$  for the unit balls of radius 1 centred at the vertices of a regular simplex inscribed in  $S^{n-1}$ , if  $\varepsilon > 0$  is small enough then the balls  $E_i = (1 - \varepsilon)B_i$  have diameter  $2 - 2\varepsilon < 2$  and cover the entire unit ball  $B^n$  with boundary  $S^{n-1}$ .)

Setting  $F_i = f^{-1}(E_i)$ ,  $i = 1, \dots, n + 1$ , we obtain a cover of  $S^n$  by the closed sets  $F_1, \dots, F_{n+1}$ . By (4), one of the  $F_i$  contains antipodal points  $x_0$  and  $-x_0$ , i.e.,  $f(x_0), f(-x_0) \in E_i$ . As  $E_i$  does not contain a pair of antipodal points,  $f(x_0)$  and  $f(-x_0)$  are not antipodal, so the map  $f$  is not antipodal either.  $\square$

**Notes.** The assertions above are classical and very well known results in general and algebraic topology, and can be found in most books on the subject; see, e.g., Dugundji or Spanier. As we noted when posing the problem, these results follow easily from *Borsuk's theorem* that continuous antipodal maps  $S^n \rightarrow S^n$  have odd degrees (see pp. 189–190 of Massey's book); in particular, they are not null-homotopic. In Borsuk's original paper on *three theorems on the  $n$ -dimensional Euclidean sphere*, the first theorem was this assertion, and the other two were assertions (2) and (4).

Any one of these equivalent assertions is usually referred to as *Borsuk's theorem* or *Borsuk's antipodal theorem*. In addition, somewhat surprisingly, assertion (2) is also called the *Borsuk–Ulam theorem*. The reason why Ulam is also given credit for this result is Borsuk's footnote in his paper that assertion (2) had been conjectured by Ulam. The fourth assertion was first proved by Lusternik and Schnirelmann (by a combinatorial method, without any appeal to algebraic topology!), and is frequently called the *Lusternik–Schnirelmann–Borsuk theorem*. The original formulation was in terms of arbitrary sets: the sphere  $S^n$  cannot be partitioned into  $n + 1$  sets, each of diameter strictly less than 2. Since the closure of a set has the same diameter as the set itself, as the very first step of our proof, we may replace the sets by their closures.

Lusternik and Schnirelmann were from the Soviet Union, so their names are transliterations from the Cyrillic; in English Schnirelmann is better spelled as Shnirelman (although the somewhat artificial spelling Shnirel'man is closer to the original Russian spelling). I have chosen to spell his name with sch and double n since he himself spelt it that way, even when writing in French, in particular, in the relevant pamphlet.

Needless to say, in these results  $S^n$  may be replaced by a space homeomorphic to  $S^n$ , with an involution coming from the antipodality in  $S^n$ . In the simplest case, instead of  $S^n$ , the unit sphere of  $\ell_2^{n+1}$ , the  $(n + 1)$ -dimensional Euclidean space, we may take the unit sphere of any  $(n + 1)$ -dimensional normed space, e.g., the unit

sphere

$$S(\ell_\infty^{n+1}) = \left\{ x = (x_i)_1^{n+1} : \max_{1 \leq i \leq n+1} |x_i| = 1 \right\},$$

with  $x$  and  $-x$  antipodal.

For combinatorial proofs and generalizations of the Borsuk–Ulam–Lusternik–Schnirelmann results, see the papers of Tucker, Jaworowski and Moszyński, Rattray, Ky Fan, Weiss, Fréchet and Fan, Prescott and Su, and many others. For example, Jaworowski, Moszyński and Rattray prove that if  $f : S^n \rightarrow S^n$  is a continuous map such that  $f(-x) \neq -f(x)$  for every  $x \in S^n$ , then there is a point  $x_0 \in S^n$  such that  $f(-x_0) = f(x_0)$ .

K. Borsuk, Drei Sätze über die  $n$ -dimensionale euklidische Sphäre, *Fund. Math.* **20** (1933), 177–190.

J. Dugundji, *Topology*, Allyn and Bacon, (1966), xvi + 447 pp.; pp. 347–350.

K. Fan (Ky Fan), A generalization of Tucker’s combinatorial lemma with topological applications, *Ann. of Math.* **56** (1952), 431–437.

K. Fan (Ky Fan), Antipodal and fixed point theorems for sets in  $\mathbb{R}^n$  bounded by a finite number of spheres, *Advances in Applied Mathematics* **22** (1999), 219–226.

M. Fréchet and K. Fan, *Invitation to Combinatorial Topology*, translated from the French, with notes, by Howard W. Eves. Reprint of the 1967 English translation. Dover Publications, Inc., (2003), xii + 124 pp.

J.W. Jaworowski and K. Moszyński, A theorem on mappings of the sphere into the projective space, *Bull. Acad. Polon. Sci. Cl. III.* **4** (1956), 75–77.

L. Lusternik and L. Schnirelmann, *Méthodes Topologiques dans les Problèmes Variationelles* (in Russian, with a French subtitle, as above), Issledovatel’kii Institut Matematiki i Mekhaniki pri O.M.G.U., Moscow, 1930, pp. 68.; for the antipodal covering theorem, see pp. 26–31.

W. Massey, *A Basic Course in Algebraic Topology*, Graduate Texts in Mathematics **127**, Springer-Verlag, (1991), xvi + 428 pp.

T. Prescott and F.E. Su, A constructive proof of Ky Fan’s generalization of Tucker’s lemma, *J. Combin. Theory Ser. A* **111** (2005), 257–265.

B.A. Rattray, Generalizations of the Borsuk–Ulam theorem, *1958 Symposium Internacional de Topología Algebraica* (International Symposium on Algebraic Topology), pp. 302, Universidad Nacional Autónoma de México and UNESCO, Mexico City.

A.W. Tucker, Some topological properties of disk and sphere, in *Proc. First Canadian Math. Congress*, Montreal, 1945, University of Toronto Press, (1946), pp. 285–309.

E.H. Spanier, *Algebraic Topology*, corrected reprint, Springer-Verlag, (1981), xvi + 528 pp.

B. Weiss, A combinatorial proof of the Borsuk–Ulam antipodal point theorem, *Israel J. Math.* **66** (1989), 364–368.

## 42. Bodies of Diameter 1: Borsuk's Problem

Let  $K \subset \mathbb{R}^d$  be a body of diameter 1. Then

- (i)  $K$  is contained in a (closed) ball of radius  $r = \sqrt{d/(2d+2)}$ ;
- (ii)  $K$  can be partitioned into  $2^{d-1} + 1$  sets, each of diameter strictly less than 1.

*Proof.* (i) We may and shall assume that  $K$  is a closed set, and the ball of minimal radius containing  $K$ ,  $B_\rho$ , has radius  $\rho$  and is centred at the origin. Let  $S_\rho = \partial B_\rho$  be the sphere bounding this ball. Since  $K$  is closed and  $\rho$  is minimal, the origin is in the convex hull of  $K \cap S_\rho$ ; consequently, by Carathéodory's theorem (see Problem 28 (ii)), there are points  $\mathbf{a}_1, \dots, \mathbf{a}_m \in K \cap S_\rho$ ,  $m \leq n + 1$ , and strictly positive reals  $\lambda_1, \dots, \lambda_m$  such that

$$\sum_{i=1}^m \lambda_i \mathbf{a}_i = 0.$$

With  $\langle \cdot, \cdot \rangle$  denoting the standard inner product, let  $\min_{1 \leq i, j \leq m} \langle \mathbf{a}_i, \mathbf{a}_j \rangle = -\varepsilon \rho^2$ . Then

$$0 = \left\| \sum_{i=1}^m \lambda_i \mathbf{a}_i \right\|^2 = \sum_{i=1}^m \lambda_i^2 \langle \mathbf{a}_i, \mathbf{a}_i \rangle + 2 \sum_{i < j} \lambda_i \lambda_j \langle \mathbf{a}_i, \mathbf{a}_j \rangle \geq \rho^2 \left( \sum_{i=1}^m \lambda_i^2 - 2\varepsilon \sum_{i < j} \lambda_i \lambda_j \right).$$

Hence,

$$2\varepsilon \sum_{i < j} \lambda_i \lambda_j \geq \sum_{i=1}^m \lambda_i^2,$$

so, by Cauchy's inequality,

$$\varepsilon m \left( \sum_{i=1}^m \lambda_i^2 \right) \geq \varepsilon \left( \sum_{i=1}^m \lambda_i \right)^2 = \varepsilon \sum_{i=1}^m \lambda_i^2 + 2\varepsilon \sum_{i < j} \lambda_i \lambda_j \geq (1 + \varepsilon) \sum_{i=1}^m \lambda_i^2.$$

Therefore,  $m\varepsilon \geq 1 + \varepsilon$ , i.e.,  $\varepsilon \geq 1/(m-1) \geq 1/n$ .

Assuming, as we may, that the minimum of  $\langle \mathbf{a}_i, \mathbf{a}_j \rangle$  is attained at  $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle$ , so that  $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = -\varepsilon \rho^2 \leq -\rho^2/n$ , we find that

$$1 \geq d(\mathbf{a}_1, \mathbf{a}_2)^2 = \|\mathbf{a}_1 - \mathbf{a}_2\|^2 = 2\rho^2 - 2\langle \mathbf{a}_1, \mathbf{a}_2 \rangle \geq \left(2 + \frac{2}{n}\right) \rho^2.$$

Hence  $\rho \leq \sqrt{n/(2n+2)} = r$ , as required.

(ii) Clearly,  $n$  orthogonal hyperplanes partition a ball of radius  $r$  in  $\mathbb{R}^n$  into  $2^n$  sets of diameter  $\sqrt{2}r$ . Consequently, by part (i), our set  $K$  can be partitioned into  $2^n$



sets of diameter at most  $\sqrt{2}r = \sqrt{n/(n+1)} < 1$ . To prove that  $2^{n-1} + 1$  parts will do, we have to do a little work.

The idea is very simple. As some point  $\mathbf{c}$  of  $B_r$  may be assumed to belong to  $K$ , the entire body  $K$  is contained in the intersection  $C \cap B_r$  of the unit ball  $C$  centred at  $\mathbf{c}$  and the smaller ball  $B_r$ . If this intersection were ‘flat’ enough, so that it would ‘resemble’ a ball of radius  $r$  in  $n - 1$  dimensions, then we could easily cut it into  $2^{n-1}$  sets of diameter substantially less than 1 by using orthogonal hyperplanes. Unfortunately,  $C \cap B_r$  is not flat enough for this purpose. What saves us is that we can cut a small set (of diameter less than 1, or even as small as we like) off the top of the smaller ball  $C$  (the part around  $\mathbf{c}$ ), as if it were a soft-boiled egg: this single small cut flattens the rest sufficiently that our orthogonal hyperplanes will do the job.

To carry out this plan, first we write  $\mathbf{c} = (0, 0, \dots, 0, r)$  for the ‘North Pole’ of the ball  $B_r = \{\mathbf{a} \in \mathbb{R}^n : \|\mathbf{a}\| \leq r\}$ , and assume, as we may, that this point  $\mathbf{c}$  belongs to  $K$ , so that  $\mathbf{c} \in K \subset B_r$ . Since  $K$  has diameter 1, it is also contained in the ball  $C$  of radius 1 and centre  $\mathbf{c}$ , and so  $K \subset L = B_r \cap C$ . We shall show that this ‘lens’  $L$  can be partitioned into  $2^{n-1} + 1$  sets of diameter less than 1.

Fix  $0 < \varepsilon < r$ , and set

$$u_\varepsilon = \sqrt{1 - \frac{n+1}{2n}\varepsilon^2} < 1,$$

and

$$d_\varepsilon = \sqrt{\frac{n+u_\varepsilon}{n+1}} < 1.$$

The reason for our choice of  $d_\varepsilon$  is that

$$d_\varepsilon^2 = 2r^2 + 2h\ell,$$

where  $h = \sqrt{r^2 - \varepsilon^2/4} > 0$  and  $\ell = 1/(2r) - r = \sqrt{\frac{n+1}{2n}} - \sqrt{\frac{n}{2(n+1)}} > 0$ .

**Claim.** *The lens  $L$  can be partitioned into a set of diameter  $\varepsilon$  and  $2^{n-1}$  sets of diameter  $d_\varepsilon$ .*

To prove this claim, write  $H$  for the half-space  $\{\mathbf{x} \in \mathbb{R}^n : x_n \leq h\}$ , with  $h$  as above, and set  $D_0 = L \setminus H$ . Then  $\text{diam}(D_0) = 2\sqrt{r^2 - h^2} = \varepsilon$ . Cutting the rest of  $L$  into  $2^{n-1}$  congruent sets by the canonical orthogonal hyperplanes  $x_1 = 0, x_2 = 0, \dots, x_{n-1} = 0$ , all we have to show is that one of these sets, say,

$$L^+ = \{\mathbf{x} = (x_1, \dots, x_n) \in L : x_i \geq 0 \text{ for } i = 1, \dots, n-1, \text{ and } x_n \leq h\}$$

has diameter at most  $d_\varepsilon$ .

Let then  $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n) \in L^+$ . To bound the distance  $d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|$ , we consider two cases according to  $\min\{a_n, b_n\}$ .

First, suppose that  $\min\{a_n, b_n\} \geq -\ell$ , where, as above,  $\ell = 1/(2r) - r > 0$ . Since  $-\ell \leq a_n, b_n \leq h$ , we find that  $a_nb_n \geq -\ell h$ . Consequently,

$$\begin{aligned} d^2(\mathbf{a}, \mathbf{b}) &= \sum_{i=1}^n (a_i - b_i)^2 = \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 - 2 \sum_{i=1}^{n-1} a_i b_i - 2a_n b_n \\ &\leq 2r^2 + 2\ell h = d_\varepsilon^2. \end{aligned}$$

Second, suppose that  $\min\{a_n, b_n\} < -\ell$ . By symmetry, we may assume that  $a_n < -\ell$ . Since  $\mathbf{a} \in C$ , we have  $d^2(\mathbf{a}, \mathbf{c}) = \sum_{i=1}^{n-1} a_i^2 + (a_n - r)^2 \leq 1$ , i.e.,  $\sum_{i=1}^n a_i^2 \leq 1 + 2ra_n - r^2$ . Consequently,

$$\begin{aligned} d^2(\mathbf{a}, \mathbf{b}) &= \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 - 2 \sum_{i=1}^{n-1} a_i b_i - 2a_n b_n \\ &\leq (1 + 2ra_n - r^2) + r^2 - 2a_n b_n = 1 + 2a_n(r - b_n) \\ &< 1 - 2\ell(r - h) = 2r^2 + 2\ell h = d_\varepsilon^2. \end{aligned}$$

The last inequality above follows from  $a_n < -\ell$  and  $r - b_n \geq r - h$ , and the penultimate equality holds since  $-2\ell r = 2r^2 - 1$ . This completes the proof of our Claim, and so our result is proved.  $\square$

**Notes.** The problem of decomposing a set of diameter 1 into sets of diameter strictly less than 1 was posed by Borsuk in 1933. Writing  $k(n)$  for the minimal integer  $k$  such that every set of diameter 1 in  $\mathbb{R}^n$  is the union of  $k$  sets of diameter less than 1, Borsuk wondered whether  $k(n)$  was equal to  $n + 1$  for every  $n \geq 1$ . (The  $n$ -dimensional simplex shows that  $k(n) \geq n + 1$  for every  $n$ .)

*Borsuk's conjecture*, as this problem became known, has attracted the attention of numerous mathematicians for over seventy years, but the determination of  $k(n)$  is far from complete. It is known that  $k(n)$  is indeed  $n + 1$  for  $n = 1$  (trivial),  $n = 2$  (shown in Problem 18) and in 1955 Eggleston proved the 'conjecture' for  $n = 3$ . A little later, Grünbaum and Heppes independently found a considerably simpler proof. The idea is to proceed along the lines of the solution of Problem 32: strengthening a result of Gale, it can be shown that every three-dimensional body of diameter 1 is contained in a regular octahedron of diameter  $\sqrt{3}$  with three of its corners truncated, and then it is easy to decompose this truncated octahedron into four sets of diameter less than 1.

Concerning a body  $K$  in dimension  $n$ , extending a result of Hadwiger, Lenz proved that if every boundary point of  $K$  is *regular*, i.e., lies on a unique supporting hyperplane, then  $K$  can be partitioned into  $n + 1$  sets of diameter less than 1. Although this result of Hadwiger may give the impression that  $k(n)$  can hardly be bigger than  $n + 1$ , progress with the problem was slow. Surprising most 'experts',

over fifty years after the problem was posed, Kahn and Kalai proved that, amazingly,  $k(n)$  is not equal to  $n + 1$  for large values of  $n$ : in fact, it grows exponentially. We shall present this beautiful result of Kahn and Kalai in Problem 133.

Lenz noticed that  $k(n) < q_n^n$ , where  $q_n = \lfloor \sqrt{2n} + 1 \rfloor$  is the smallest integer strictly greater than  $\sqrt{2n}$ . Indeed, every set of diameter 1 is contained in a cube of side-length 2, and every cube of side-length 2 can be decomposed into  $q_n^n$  subcubes of side-length less than  $1/\sqrt{n}$  (and so of diameter less than 1). This super-exponential bound of Lenz was improved by Borsuk, and then by Lassak to the bound in this question: the proof we present is exactly his. The very simple first part of the question is a result Jung proved over a hundred years ago.

Lassak's bound was improved by Schramm to  $5n\sqrt{n}(4 + \log n)(\frac{3}{2})^{n/2}$ . Bourgain and Lindenstrauss proved a similar bound on a function closely related to  $k(n)$ : if  $\varepsilon > 0$  and  $n$  is sufficiently large then any set of diameter 1 in dimension  $n$  can be covered by no more than  $(\frac{3}{2} + \varepsilon)^{n/2}$  Euclidean balls of diameter 1.

K. Borsuk, Drei Sätze über die  $n$ -dimensionale euklidische Sphäre, *Fund. Math.* **20** (1933), 177–190.

K. Borsuk, Some remarks on covering of bounded subsets of the Euclidean  $n$ -space with sets of smaller diameter, *Demonstratio Math.* **11** (1978), 247–251.

J. Bourgain and J. Lindenstrauss, On covering a set in  $\mathbb{R}^N$  by balls of the same diameter, in *Geometric Aspects of Functional Analysis* (1989–90), Lecture Notes in Math., **1469**, Springer, (1991), pp. 138–144.

H.G. Eggleston, Covering a three dimensional set with sets of smaller diameter, *J. London Math. Soc.* **30** (1955), 11–24.

D. Gale, On inscribing  $n$ -dimensional sets in a regular  $n$ -simplex, *Proc. Amer. Math. Soc.* **4** (1953), 222–225.

B. Grünbaum, A simple proof of Borsuk's conjecture in three dimensions, *Proc. Cambridge Philos. Soc.* **53** (1957), 776–778.

B. Grünbaum, Borsuk's problem and related questions, in *Convexity*, Proc. Symposia in Pure Maths, vol. VII., American Math. Soc., (1963), pp. 271–284.

H. Hadwiger, Über die Zerstückung eines Eikörpers, *Math. Zeitschrift* **51** (1948), 161–165.

A. Heppes, On partitioning of three-dimensional point-sets into sets of smaller diameter (in Hungarian), *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.* **7** (1957), 413–416.

H.W.E. Jung, Über die kleinste Kugel, die eine räumliche Figur einschliesst, *J. Reine Angew. Math.* **123** (1901), 241–257.

H. Lenz, Zur Zerlegung von Punktmengen in solche kleineren Durchmessers, *Arch. Math.* **6** (1955), 413–416.

M. Lassak, An estimate concerning Borsuk partition problem, *Bull. Acad. Polon. Sci. Sér. Sci. Math.* **30** (1982), 449–451.

O. Schramm, Illuminating sets of constant width, *Mathematika* **35** (1988), 180–189.

### 43. Equilateral Triangles: Napoleon's Theorem

*If equilateral triangles are erected externally (or internally) on the sides of a triangle then the centroids of the erected triangles form an equilateral triangle whose centroid is the centroid of the original triangle.*

*Proof.* This is one of the first exercises in the use of vectors and complex numbers for 'epsilons' – unfortunately, after the slogan 'let's use vectors and complex numbers' no more thinking is needed.

Let the triangles  $ABC$  and  $A'B'C'$  be as in Figure 51, with vectors (complex numbers)  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$ , and  $\omega = e^{i\pi/3}$ , so that  $\omega^3 = -1$  and  $\omega^2 + 1 = \omega$ . Thus,  $\overrightarrow{AB} = (1 + \omega)\mathbf{c}$ ,  $\overrightarrow{BC} = (1 + \omega)\mathbf{a}$  and  $\overrightarrow{CA} = (1 + \omega)\mathbf{b}$ ; similarly,  $\overrightarrow{A'B'} = \omega\mathbf{a} + \mathbf{b}$ ,  $\overrightarrow{B'C'} = \omega\mathbf{b} + \mathbf{c}$  and  $\overrightarrow{C'A'} = \omega\mathbf{c} + \mathbf{a}$ . Since  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = (1 + \omega)(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$  we have  $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ . Note that  $A'B'C'$  is an equilateral triangle if and only if

$$\omega \overrightarrow{A'B'} = \overrightarrow{A'C'}.$$

But

$$\omega \overrightarrow{A'B'} - \overrightarrow{A'C'} = \omega(\omega\mathbf{a} + \mathbf{b}) + (\omega\mathbf{c} + \mathbf{a}) = (\omega^2 + 1)\mathbf{a} + \omega\mathbf{b} + \omega\mathbf{c} = \omega(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0},$$

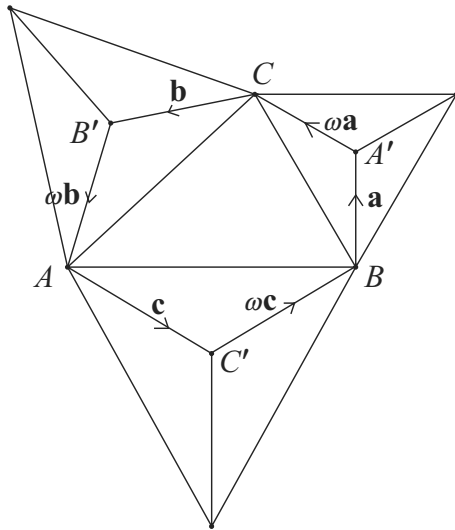


Fig. 51. Three equilateral triangles erected on a triangle  $ABC$ , with centroids  $A'$ ,  $B'$  and  $C'$ ,  $\overrightarrow{BA'} = \mathbf{a}$ ,  $\overrightarrow{AC'} = \omega\mathbf{a}$ , and so on.

so  $A'B'C'$  is indeed an equilateral triangle. The second assertion follows from a similar trivial calculation.  $\square$

**Notes.** Much of the attraction of this question comes from its very dubious association with Napoleon Bonaparte. In fact, the earliest reference to this result seems to have been made in 1825 by W. Rutherford, although it is probably older. As Coxeter and Greitzer noted, “the possibility of [Napoleon] knowing enough geometry for this feat is as questionable as the possibility of his knowing enough English to compose the famous palindrome, ABLE WAS I ERE I SAW ELBA.”

The result has countless generalizations, extensions, and converses, many of which are highly non-trivial. For a host of references, see Martini.

H.S.M. Coxeter, *Introduction to Geometry*, Reprint of the 1969 edition, Wiley Classics Library, John Wiley & Sons, Inc., New York, (1989), xxii + 469 pp.

H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, New Math. Library **19**, Math. Assoc. of America, (1967).

S.B. Gray, Generalizing the Petr–Douglas–Neumann theorem on  $n$ -gons, *Amer. Math. Monthly* **110** (2003), 210–227.

H. Martini, On the theorem of Napoleon and related topics, *Math. Semesterber.* **43** (1996), 47–64.

C.J. Scriba, Wie kommt “Napoleons Satz” zu seinem Namen? *Historia Math.* **8** (1981), 458–459.

A.R. (Alvy Ray) Smith, Infinite regular hexagon sequences on a triangle, *Experiment. Math.* **9** (2000), 397–406.

J.E. Wetzel, Converses of Napoleon’s theorem, *Amer. Math. Monthly* **99** (1992), 339–351.

## 44. Trisectors of Angles: Morley's Theorem

*The trisectors of the angles of a triangle meet at the vertices of an equilateral triangle.*

*Proof.* Let  $PQR$  be the triangle formed by the trisectors of a triangle  $ABC$ , with angles  $3\alpha$ ,  $3\beta$  and  $3\gamma$ , so that  $\alpha + \beta + \gamma = \pi/3$ , as in Figure 52.

For simplicity, let us write  $x^+$  for  $x + \pi/3$ . We claim that the angles of six of the triangles are as in Figure 52. Note that the assignment of angles is consistent with the requirements that the sum of angles in a triangle is  $\pi$ . For example, in the triangle  $ARQ$  we have  $\alpha + \beta^+ + \gamma^+ = (\alpha + \beta + \gamma) + 2\pi/3 = \pi$ . However, it is far from obvious that this is indeed the distribution of angles. Nevertheless, the assertion that the angles are distributed as claimed is more than the assertion we have to prove since if this is the distribution then the angle  $QPR$  is  $2\pi - (\alpha^{++} + \beta^+ + \gamma^+) = \pi/3$ , and the angles  $PRQ$  and  $RQP$  are also  $\pi/3$ .

We do know that the angle  $BPC$  is  $\pi - (\beta + \gamma) = \alpha^{++}$ ; similarly,  $\beta^{++}$  and  $\gamma^{++}$  appear as claimed. What is not clear is that  $\alpha^+$ ,  $\beta^+$  and  $\gamma^+$  appear as claimed. Let us see two ways of completing the solution.

*1. Brute force.* We can proceed mindlessly, attacking the problem head on with a little trigonometry. We may assume that  $BC = \sin 3\alpha$ ,  $CA = \sin 3\beta$  and  $AB = \sin 3\gamma$ . Consequently,

$$AR = \frac{\sin \beta}{\sin \gamma^{++}} \sin 3\gamma \text{ and } AQ = \frac{\sin \gamma}{\sin \beta^{++}} \sin 3\beta.$$

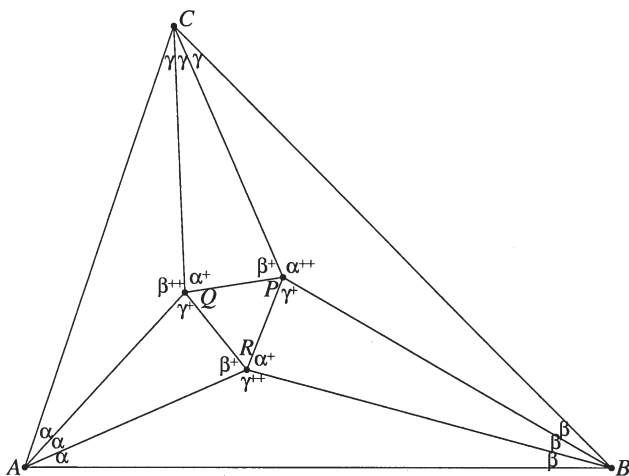


Fig. 52. The distribution of angles, with  $x^+ = x + \pi/3$ .

Hence, the angles in the triangle  $ARQ$  are as claimed if  $AQ/AR = \sin \beta^+ / \sin \gamma^+$ , i.e.,

$$\sin 3\beta \sin \gamma \sin \gamma^+ \sin \gamma^{++} = \sin 3\gamma \sin \beta \sin \beta^+ \sin \beta^{++}.$$

This is indeed the case since

$$\sin 3x = 4 \sin x \sin x^+ \sin x^{++} = 3 \sin x - 4 \sin^3 x.$$

2. *A little finesse.* Let us work backwards. To prove the assertion, it suffices to show that if  $PQR$  is an equilateral triangle and we erect on its sides triangles  $RQA$ ,  $PRB$  and  $QPC$  as in Figure 53 then we get a triangle  $ABC$  with angles  $3\alpha$ ,  $3\beta$ ,  $3\gamma$ , and appropriate trisectors. Let us reflect  $P$  in  $BR$  to get  $R_1$ , and  $Q$  in  $AR$  to get  $R_2$ ; construct the points  $P_1$ ,  $P_2$ ,  $Q_1$  and  $Q_2$  similarly, as in Figure 53. All we need then is that the points  $R_1$  and  $R_2$  are on  $AB$  (and so  $P_1$ ,  $P_2$ ,  $Q_1$  and  $Q_2$  are also on the appropriate sides). This is easily seen by computing some angles. For example,  $ARR_2$  is  $\beta^+$ , so  $ARR_1$  is  $\beta^+ - (\pi - 2\gamma^+)$ . Let us assume that  $\alpha + \beta \geq \gamma$ , i.e., the points  $R_1$  and  $R_2$  are in the order as in Figure 54. Then the angle  $R_1RR_2$  is  $\beta^+ + \alpha^+ - \gamma^{++} = \alpha + \beta - \gamma = \pi/3 - 2\gamma$ , so the angle  $RR_2R_1$  is  $(\pi - (\pi/3 - 2\gamma))/2 = \gamma^+$ . This implies that the point  $R_1$  is on the segment  $AR_2$ . Similarly,  $R_2$  is on the segment  $BR_1$ , and we are done.

If  $\alpha + \beta < \gamma$  then  $R_1$  and  $R_2$  are interchanged on  $AB$ ; to see that, we note that the angle  $R_1RR_2$  is  $2\gamma - \pi/3$  and  $RR_2R_1$  is  $\pi - \gamma^+$ , so  $R_1$  is on  $BR_2$ . Similarly,  $R_2$  is on  $AR_1$ , completing the solution.  $\square$

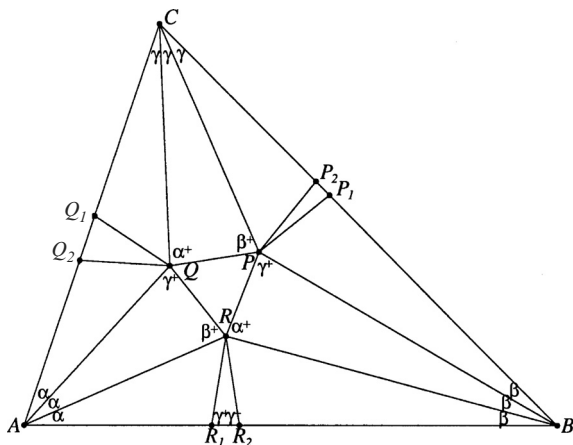


Fig. 53. Start with an equilateral triangle  $PQR$ , add the triangles  $PQC$ ,  $QRA$  and  $RPB$ , reflect  $P$  in  $BR$  to get  $R_1$  and  $Q$  in  $AR$  to get  $R_2$ , and so on.

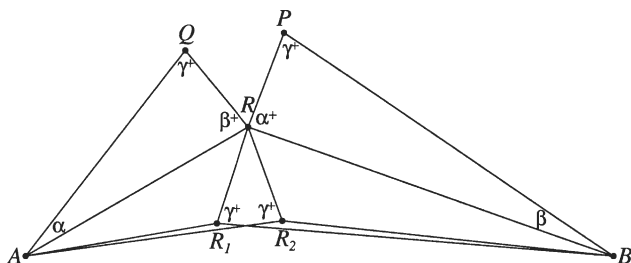


Fig. 54. The arrangement of the points  $R_1$  and  $R_2$  when  $\alpha + \beta \geq \gamma$ .

**Notes.** Morley's theorem is, needless to say, an old chestnut. Nevertheless, it is not too well known to young mathematicians. In the early 1970s, before the total demise of geometry in schools, in Cambridge I frequently set it to my better freshmen, who never found it easy. However, after the hint the light suddenly dawned, and both solutions above were found by many students.

In theory, the problem would be an ideal *experimental mathematics project* for undergraduates: having determined the angles in half a dozen examples, it is rather difficult to miss the distribution of angles ( $\alpha^+$ ,  $\beta^{++}$ , etc.) and, having guessed this, a proof should be just around the corner.



## 45. Connected Subgraphs

*A graph of maximal degree  $d \geq 3$  has at most  $(e(d-1))^n$  connected induced subgraphs with  $n+1$  vertices, one of which is a given vertex.*

*First Proof.* Let  $G^*$  be a *rooted graph*, i.e., a graph  $G$  with a specified vertex, its *root*, and let  $d \geq 3$  be the maximal degree of  $G^*$ . Our task is to give an upper bound for  $C(G^*; n)$ , the number of connected induced subgraphs of  $C(G^*; n)$  with  $n+1$  vertices, one of which is the root.

As in the *Hint*, let  $N(G^*; n)$  be the number of subtrees of  $G^*$  containing the root and having  $n$  other vertices. As every connected graph has a spanning tree,

$$C(G^*; n) \leq N(G^*; n) \quad (6)$$

for every rooted graph  $G^*$  and every integer  $n$ .

We shall prove more than is required, namely that

$$N(G^*; n) \leq \binom{dn - n + 1}{n} \quad (7)$$

In fact, it is easy to see that (7) holds for  $d = 2$  as well since then rather trivially,  $N(G^*; n) \leq n + 1$ , with equality attained on all cycles with at least  $n + 2$  vertices. For  $d \geq 3$ ,

$$\binom{dn - n + 1}{n} = \frac{1}{n!} (d-1)n \left( (d-1)^2 n^2 - 1 \right) ((d-1)n - 2)_{(n-3)} \leq (e(d-1))^n,$$

so (7) is indeed more than required.

Following the *Hint*, first we show that

$$N(G^*; n) \leq N(T_d^*; n) = N(T_{d,n}^*; n), \quad (8)$$

where  $T_d$  is the Cayley tree of degree  $d$  and  $T_{d,n}^*$  is the  $d$ -regular rooted tree of height  $n$ : the unique rooted tree of height  $n$  with  $1 + d + d(d-1) + \dots + d(d-1)^{n-1}$  vertices (see Figure 55).

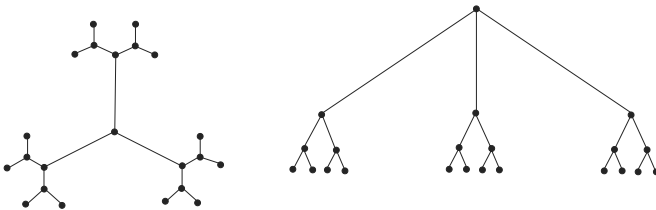


Fig. 55. The cubic Cayley tree  $T_3$  and the cubic tree of height 3,  $T_{3,3}$ .

To this end, label the edges of  $T_d^*$  as follows. First, label the edges incident with the root with  $1, 2, \dots, d$ , and for every other vertex  $x$ , label the  $d - 1$  edges joining  $x$  to its descendants ('sons') with  $1, 2, \dots, d - 1$ . Next, for every vertex  $x$  of  $G$ , order the edges incident with  $x$  linearly.

Let  $H$  be a subtree of  $G^*$  with  $n + 1$  vertices, one of which is the root. For every vertex  $x$  of  $H$  we can describe the unique path  $x_0x_1 \dots x_\ell$  in  $H$  from the root  $* = x_0$  to  $x = x_\ell$  by a sequence  $\alpha_H(x) = (a_1, a_2, \dots, a_\ell)$  by setting  $a_i = r$  if the edge  $x_i x_{i+1}$  is the  $r$ th of the edges incident with  $x_i$  other than the edge leading to  $x_i$ , namely  $x_{i-1}x_i$ . Needless to say,  $x_0x_1$  is the  $a_0$ th edge incident with  $x_0$ , as there is no  $x_{-1}x_0$  edge. Also, the root itself gets labelled with the empty sequence. Clearly, every sequence  $\alpha(x)_H = (a_0, a_1, \dots, a_{\ell-1})$  is such that  $1 \leq a_0 \leq d$  and  $1 \leq a_i \leq d - 1$  for  $i \geq 1$ . Note also that the set  $A(H)$  of the labels of the vertices of  $H$  determines  $H$ : if  $H \neq H'$  then  $A(H) \neq A(H')$ .

A map  $x \rightarrow \beta(x)$  similar to  $\alpha$  can be applied to the vertices of  $T_{d,n}^*$  (or  $T_d^*$ ) as well, but this time without any reference to a subtree  $H$ : for every vertex  $x$  of  $T_{d,n}^*$  there is a unique path  $x_0x_1 \dots x_\ell$ , where  $x_0$  is the root of  $T_d^*$  and  $x_\ell = x$ , and so on. Clearly, the map  $\beta$  gives a one-to-one correspondence between the vertex set of  $T_d^*$  and the set  $W_d$  of all finite integer sequences  $(b_0, b_1, \dots, b_\ell)$  such that  $1 \leq b_0 \leq d$ ,  $1 \leq b_i \leq d - 1$  for every  $i \geq 1$ , and  $\ell = -1, 0, 1, \dots$  (For  $\ell = -1$  we have the empty sequence.) Also,  $\beta$  gives a correspondence between the vertex set of  $T_{d,n}^*$  and the subset  $W_{d,n}$  of  $W_d$  consisting of all sequences of length at most  $n$ .

So far, we have done extremely little, although we have taken our time over it: we have defined a numbering of the vertex set of the entire Cayley tree  $T_d^*$  and also the vertex set of every subtree  $H$  of  $G$  containing the root. However, this is all we need to see (8): given a subtree  $H$  of  $G$ , the set of vertices of  $T_d^*$  whose labels are in  $A(H)$  form a subtree  $\gamma(H)$  of  $T_d^*$  containing the root, with as many vertices as  $H$ . Finally, if  $H \neq H'$  then, as  $A(H) \neq A(H')$ , we have  $\gamma(H) \neq \gamma(H')$  as well, proving (8).

Turning to the heart of the question, we shall show that

$$N(T_d^*; n) \leq \binom{dn - n + 1}{n}; \quad (9)$$

by (8) this does imply (7).

For simplicity, let us identify the vertices of  $T_{d,n}^*$  with their labels introduced above, so that the vertex set  $V(T_{d,n})$  is exactly  $W_{d,n}$ . Order the elements of  $W_{d,n}$  lexicographically, i.e., set  $a = (a_0, a_1, \dots, a_k) < (b_0, b_1, \dots, b_\ell)$  if there is an index  $m \leq \min\{k, \ell\}$  such that  $a_i = b_i$  for  $1 \leq i \leq m$  and either  $m = k < \ell$  or else  $m < \min\{k, \ell\}$  and  $a_{m+1} < b_{m+1}$ . If  $a < b$  then we say that  $a$  precedes  $b$  or  $a$  is smaller than  $b$ .

Although it does not concern us, note that in the analogous lexicographic order on  $V(T_d^*)$  there are infinitely many elements between (1) and (2), say, namely

all the sequences starting with 1. Also, a non-empty sequence  $(a_0, \dots, a_\ell)$  has a maximal element preceding it if and only if  $a_\ell = 1$ , in which case  $(a_0, \dots, a_{\ell-1})$  is this maximal element.

The lexicographic order on  $W_{d,n}$  enables us to code subtrees of  $T_d^*$  containing the root in a very simple way. Let  $H$  be a subtree of  $T_{d,n}^*$  with vertex set  $\{x_0, x_1, \dots, x_n\}$ , where  $x_0 = \emptyset < x_1 < \dots < x_n$ . Since every sequence precedes each of its continuations, for  $1 \leq i \leq n$  the vertices  $x_0, x_1, \dots, x_{i-1}$  are the vertices of a subtree  $H_i$  of  $H$ . (Note that  $H_i$  has  $i$  vertices, including the root.) Let  $D_i$  be the set of descendants of these vertices that do not belong to  $H_i$ . Denote by  $h_i$  the order of  $x_i$  in  $D_i$  in the lexicographic order. (Putting it slightly differently,  $x_i$  is the  $h_i$ th element of  $D_i$ .) Clearly, the sequence  $(h_1, h_2, \dots, h_n)$  determines  $H$ , and therefore so does the set  $S = \{s_1, s_2, \dots, s_n\}$ , where  $s_j = \sum_{i=1}^j h_i$ . Note that there are at least  $s_n = \sum_{i=1}^n h_i$  vertices of  $T_d^*$  within distance 1 of  $H_n$ . Hence, writing  $t_n$  for the maximal number of vertices of  $T_d^*$  within distance 1 of a subtree of  $T_d^*$  with  $n$  vertices, we see that  $s_n \leq t_n$ , and so

$$N(T_{d,n}^*; n) \leq \binom{t_n}{n}. \quad (10)$$

In fact, as  $T_d^*$  is a  $d$ -regular tree, for every subtree  $H'$  of  $T_d^*$  of order  $n$  there are precisely  $t_n = dn - (n - 1)$  vertices within distance 1 of  $H'$ . Hence, (10) implies (9). Putting together inequalities (6), (8) and (9) we find that

$$C(G^*; n) \leq N(G^*; n) \leq N(T_d^*; n) \leq \binom{dn - n + 1}{n}, \quad (11)$$

completing our proof.  $\square$

Note that the proof of (10) applies to any tree, not just Cayley trees. Thus, if  $T^*$  is a rooted tree and  $p_n$  is the maximal number of vertices of  $T$  within distance 1 of a subtree of  $T$  that contains the root and has  $n$  vertices then

$$N(T; n) \leq \binom{p_n}{n}.$$

*Second Proof.* Let us accept from the first solution that, as implied by inequalities (6) and (8),  $C(G^*; n) \leq N(T_d^*; n)$ , so let us bound  $N(T_d^*; n)$ . In fact, we shall work with labelled subtrees: let  $L(T_d^*; n)$  be the number of subtrees of  $T_d^*$  with  $n + 1$  vertices, one of which is the root, labelled 0, 1,  $\dots$ ,  $n$ , with the root labelled with 0. Thus

$$L(T_d^*; n) = n!N(T_d^*; n).$$

Let us apply a basic combinatorial technique, *double counting*. Let  $\mathcal{L}_n$  be the set of trees with vertex set  $\{0, 1, \dots, n\}$ , so that  $\mathcal{L}_n$  has  $(n + 1)^{n-1}$  elements. For

$T \in \mathcal{L}_n$  let  $n_d(T)$  be the number of ways of embedding  $T$  into  $T_d^*$ , with 0 put into the root. Then

$$L(T_d^*; n) = \sum_{T \in \mathcal{L}_n} n_d(T).$$

All that remains is to bound  $n_d(T)$ . We shall be rather crude and give a simple uniform bound on  $n_d(T)$ , although  $n_d(T)$  is frequently much smaller, e.g., it is 0 whenever the maximal degree of  $T$  is greater than  $d$ . To give a uniform bound on  $n_d(T)$ , let us embed  $T$  vertex by vertex. We start with the vertex labelled 0: that goes into the root. Next, we embed a neighbour of 0: we have  $d$  choices. From then on, we always embed a vertex with at least one neighbour already embedded: at each stage we have at most  $d - 1$  choices. Indeed, if we are about to embed a vertex  $i$ , one of whose neighbours has already been put into a vertex  $x$  of  $T_d^*$ , then  $i$  must go into one of the neighbours of  $x$  not yet occupied by a vertex of  $T$ ; as  $x$  has at least one neighbour that is already occupied, we have at most  $d - 1$  choices, as claimed. Consequently,

$$n_d(T) \leq d(d - 1)^{n-1},$$

and so

$$N(T_d^*; n) = L(T_d^*; n)/n! \leq \frac{1}{n!} (n + 1)^{n-1} d(d - 1)^{n-1}.$$

Clearly, this inequality is considerably stronger than  $N(T_d^*; n) \leq (e(d - 1))^n$ , so our second proof is complete.  $\square$

## 46. Subtrees of an Infinite Tree

Let  $D_2$  be the infinite tree in which one vertex, the root, has degree 2, and every other vertex has degree 3. This tree  $D_2$  has  $\frac{1}{n+1} \binom{2n}{n}$  subtrees with  $n$  vertices, one of which is the root.

*Proof.* We shall use the generating function of the sequence in question. Thus, for  $n \geq 1$  let  $f_n$  be the number of subtrees of  $D_2$  containing  $n$  vertices, one of which is the root (see Figure 56), and set  $f_0 = 1$ . (If we ever considered the graph without any vertices or edges, then we would have to take  $f_0$  to be 1 anyhow.) Furthermore, set

$$f(X) = 1 + \sum_{n=1}^{\infty} f_n X^n.$$

In order to find an equation satisfied by  $f$ , write  $f(\ell, r)$  for the number of subtrees of  $D_2$  containing the root, with  $\ell$  vertices on the left-hand branch and  $r$  vertices on the right-hand branch (see Figure 56).

Then  $f(\ell, r) = f_\ell f_r$  and so

$$f_{n+1} = \sum_{\ell=0}^n f(\ell, n-\ell) = \sum_{\ell=0}^n f_\ell f_{n-\ell},$$

i.e.,

$$f(X) = 1 + Xf^2(X).$$

Consequently,

$$f(X) = \frac{1 - (1 - 4X)^{-1/2}}{2X} = \frac{1}{n+1} \binom{2n}{n},$$

as claimed. □

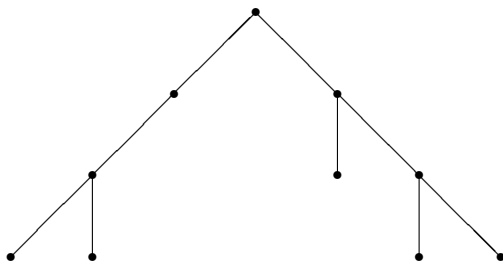


Fig. 56. A subtree of  $D_2$  with ten vertices: the root, four vertices on the left and five on the right.

## 47. Two-distance Sets

Let  $S \subset \mathbb{R}^n$  be a set of points in  $\mathbb{R}^n$  determining only two different distances. Then  $|S| \leq (n+1)(n+4)/2$ .

*Proof.* This is a lovely result of Larman, Rogers and Seidel, who also gave the elegant proof indicated in the *Hint*. In fact, the *Hint* gives away too much: very little effort is needed to complete the solution.

To recap, let  $S \subset \mathbb{R}^n$  be a *two-distance set* in  $\mathbb{R}^n$ , i.e., a set of points such that  $\|\mathbf{s} - \mathbf{s}'\| = \alpha$  or  $\beta$ , whenever  $\mathbf{s}, \mathbf{s}' \in S$ ,  $\mathbf{s} \neq \mathbf{s}'$ . Our aim is to show that  $|S| \leq (n+1)(n+4)/2$ .

For  $\mathbf{s} \in S$  define a function  $f_{\mathbf{s}} : S \rightarrow \mathbb{R}$  by

$$f_{\mathbf{s}}(\mathbf{x}) = (\|\mathbf{x} - \mathbf{s}\|^2 - \alpha^2)(\|\mathbf{x} - \mathbf{s}\|^2 - \beta^2),$$

where  $\|\mathbf{s} - \mathbf{t}\|$  is the distance between  $s$  and  $t$ . Let  $\mathcal{F}_S$  be the set of these functions, and let  $d_S$  be the dimension of the subspace of the vector space of functions  $S \rightarrow \mathbb{R}$  (over  $\mathbb{R}$ ) spanned by  $\mathcal{F}_S$ .

Clearly,  $\mathcal{F}_S$  is an independent set since for  $\mathbf{s}, \mathbf{x} \in S$ ,  $f_{\mathbf{s}}(\mathbf{x}) \neq 0$  if, and only if,  $\mathbf{x} \neq \mathbf{s}$ . Therefore  $d_S = |S|$ , and so to prove the result we have to show that

$$d_S \leq \frac{1}{2}(n+1)(n+4). \quad (12)$$

To this end, note that with  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{s} = (s_1, \dots, s_n)$ , the function  $f_{\mathbf{s}}(\mathbf{x})$  can be written as follows:

$$\begin{aligned} f_{\mathbf{s}}(\mathbf{x}) &= \|\mathbf{x}\|^4 + \sum_{1 \leq i \leq n} (-4s_i) \|\mathbf{x}\|^2 x_i \\ &\quad + \sum_{1 \leq i < j \leq n} (8s_i s_j) x_i x_j + \sum_{1 \leq i \leq n} (4s_i^2 + 2\|\mathbf{s}\|^2 - \alpha^2 - \beta^2) x_i^2 \\ &\quad + \sum_{1 \leq i \leq n} 2(\alpha^2 + \beta^2 - 2\|\mathbf{s}\|^2) s_i x_i + (\|\mathbf{s}\|^2 - \alpha^2)(\|\mathbf{s}\|^2 - \beta^2). \end{aligned}$$

Hence each element of  $\mathcal{F}_S$  belongs to the linear span of the following  $\frac{1}{2}(n+1)(n+4)$  elements of  $\mathbb{R}^S$ :  $\|\mathbf{x}\|^4$  and the identically 1 function, together with  $\|\mathbf{x}\|^2 x_i$ ,  $x_i x_j$  and  $x_i$  for  $1 \leq i \leq j \leq n$ . In particular,  $d_S \leq \frac{1}{2}(n+1)(n+4)$ , completing the proof of (12).  $\square$

**Notes.** The problem of giving bounds on the number of points in a two-distance set is probably due to Kelly, who also showed that in two dimensions there can be at most five points, realized by a regular pentagon. Croft showed that in

three dimensions there can be at most six points, realized by the midpoints of the sides of a regular simplex, i.e., by a regular octahedron. Delsarte, Goethals and Seidel gave the bound  $1 + n(n + 3)$  in  $n$  dimensions, and then Larman, Rogers and Seidel proved the bound  $\frac{1}{2}(n + 1)(n + 4)$  of this problem.

From below, the  $\frac{1}{2}n(n + 1)$  midpoints of the sides of a regular simplex in  $n$  dimensions form a two-distance set, and Delsarte, Goethals and Seidel gave examples of two-distance sets (with all points on a sphere) with  $\frac{1}{2}n(n + 3)$  points, provided  $n = 2, 6$  or  $22$ .

H.T. Croft, 9-point and 7-point configurations in 3-space, *Proc. London Math. Soc.* **12** (1962), 400–424.

P. Delsarte, J.M. Goethals and J.J. Seidel, Spherical codes and designs, *Geometriae Dedicata* **6** (1977), 363–388.

L.M. Kelly, Elementary problems and solutions, Isosceles  $n$ -points, Solution of Problem E735, posed by Paul Erdős, *Amer. Math. Monthly* **54** (1947), 227–229.

L.M. Kelly and W.O.J. Moser, On the number of ordinary lines determined by  $n$  points, *Canad. J. Math.* **1** (1958), 210–219.

D.G. Larman, C.A. Rogers and J.J. Seidel, On two-distance sets in Euclidean space, *Bull. London Math. Soc.* **9** (1977), 261–267.

## 48. Gossiping Dons

*Each of  $n \geq 4$  Oxford dons has an item of gossip known only to himself. Whenever a don telephones another, they exchange all items of gossip they know. The minimal number of calls the dons have to make in order to ensure that every one of them knows all the gossip there is to know is  $2n - 4$ .*

*Proof.* Writing  $f(n)$  for the minimal number of calls, it is trivial that  $f(1) = 0$  and  $f(2) = 1$ , and it is very easy to see that  $f(3) = 3$  and  $f(4) = 4$ . To see that  $f(n) \leq 2n - 4$  for  $n \geq 4$ , note first that 12, 34, 13, 24 is a suitable system of calls for  $n = 4$ , so  $f(4) \leq 4$ . Here, and in what follows, we denote the dons by  $1, 2, \dots, n$  and the calls by  $ij$ . With a slight abuse of notation, we also write  $i$  for the item of gossip  $i$  knows. Thus 12, 34, 13, 24 denotes the system in which first dons 1 and 2 talk, then 3 and 4, then 1 and 3 and, finally, 3 and 4. Also,  $f(n + 1) \leq f(n) + 2$  for every  $n$ , since a suitable system of  $k$  calls among  $1, 2, \dots, n$  can be extended to a suitable system of  $k + 2$  calls among  $1, 2, \dots, n, n + 1$  as follows: first  $n + 1$  talks to 1, then the first  $n$  dons make  $k$  calls to ensure that they all know their own items of gossip (and, incidentally, that of  $n + 1$  as well, since that item propagates with the item that 1 knows), and the system concludes by  $n + 1$  calling any of the



Fig. 57. Still gossiping.



other dons. (For  $n = 5$  we may take the calling scheme 51, 12, 34, 13, 24, 51. Note that the call 51 (which is the same as 15) occurs twice in the system: these are, of course, different calls.) Hence,  $f(n) \leq 2n - 4$  for  $n \geq 4$ , as claimed.

Let us turn to the real assertion that  $2n - 4$  calls are needed. Suppose for a contradiction that this is not the case, and let  $n \geq 5$  be the minimal number for which there is a suitable scheme  $S$  of  $k = 2n - 5$  calls.

To start our argument, we prove the ‘*nohoi*’ principle that in this scheme  $S$  ‘no one hears his own item’. Indeed, if *nohoi* fails then there are calls  $ax_1, x_1x_2, \dots, x_{s-1}x_s, x_s a$  made in this order. We claim that two fewer calls suffice to spread the information in the system without don  $a$ . To see this, we change the calling scheme as follows. We omit the calls  $ax_1$  and  $ax_s$  made in this sequence, rearrange the other calls involving  $a$ , and keep all other calls (in the order they were made, of course). This rearrangement goes as follows: a call  $ab$  made before  $ax_1$  is replaced by  $bx_1$  (at the very time  $ab$  was made), a call  $ab$  made after call  $x_{t-1}x_t$  and before  $x_tx_{t+1}$  is replaced by  $bx_t$ , and a call  $ab$  made after  $x_{s-1}x_s$  is replaced by  $bx_s$ . A moment’s thought tells us that this new scheme of  $k - 2$  calls works for the  $n - 1$  dons involved; hence,  $k - 2 \geq 2(n - 1) - 4$ , contradicting our assumption that  $k = 2n - 5$ .

Let us note some easy consequences of the *nohoi* principle. First of all, no call is repeated, so the set of calls forms a graph rather than a multigraph. More importantly, if  $ab$  is the last call  $a$  makes then it is also the last call  $b$  makes. Indeed, if  $ab$  is the last call for  $a$ , and later  $b$  makes a call  $bc$  then, as after his last call  $a$  knows item  $c$ , in call  $bc$  don  $c$  learns item  $c$ , contradicting *nohoi*. Hence, *last calls are last calls*. Similarly, *first calls are first calls*: if  $a$ ’s first call is to  $b$  then that call is  $b$ ’s first call as well. This can be checked similarly, and is also immediate from the fact that reversing a valid scheme of calls we obtain another valid scheme of calls since reversing the order of calls  $ax_1, x_1x_2, \dots, x_{s-1}x_s, x_sb$  taking item  $a$  to don  $b$  transmits item  $b$  to don  $a$ .

We say that a call  $ab$  is *intermediate* if it is neither a first call nor a last call. Since a first call cannot be a last call (as  $n \geq 5$ ), there are  $n/2$  first calls,  $n/2$  last calls, and  $n - 5$  intermediate calls. In particular, the graph  $H$  formed by the intermediate calls has at least five components. Writing  $C(a)$  for the component of vertex (don)  $a$  in  $H$ , note that if  $a$ ’s first call is to  $b$  then the intermediate calls that propagate item  $a$  are all in components  $C(a)$  and  $C(b)$ . Similarly, if  $a$ ’s last call is to  $c$  then the intermediate calls that propagate information to  $a$  are all in components  $C(a)$  and  $C(c)$ . Hence  $H$  has at least two components that do not propagate information from  $a$  or to  $a$ . Let us write  $u(a)$  for the number of edges in these (at least two) components. Furthermore, let us write  $w(a)$  for the number of calls that do not propagate information from  $a$  or to  $a$ : as far as  $a$  is concerned, these are *wasted* calls. We have just noted that  $u(a) \leq w(a)$  for every vertex  $a$ .

Trivially,  $n - 1$  of the calls propagate information from  $a$ , and  $n - 1$  calls propagate information to  $a$ . Since by *nohoi* only a call involving  $a$  can propagate information both from  $a$  and to  $a$ , writing  $p(a)$  for the number of phone calls made by  $a$  (so that in  $H$  vertex  $a$  has degree  $p(a) - 2$ ), we find that

$$k = 2n - 5 \geq n - 1 + n - 1 - p(a) + w(a).$$

Consequently,

$$p(a) \geq 3 + w(a) \geq 3 + u(a) \geq 3; \quad (13)$$

in particular, every don makes at least three calls, and so at least one intermediate call. But then every component of  $H$  has at least one edge, and so  $u(a)$ , counting the number of edges in certain two components, is at least 2. But then (13) implies that  $p(a) \geq 5$ , and so that there are at least  $5n/2$  phone calls, contradicting our assumption that there are  $k = 2n - 5$  phone calls. This contradiction completes our proof.  $\square$

**Notes.** This is an old chestnut indeed: there is hardly any combinatorialist who does not know it. I first put it on an examples sheet in Cambridge in 1973, and it has remained a permanent fixture of various examples sheets, and a favourite problem of many undergraduates for over thirty years. The problem was first floated in 1970; not surprisingly, many solutions were offered almost immediately: Hajnal, Milner and Szemerédi; Tijdeman; Baker and Shostak; Bumby all solved the original problem. The proof above is that given by Bumby.

Numerous extensions and variants have been proved. For example, Bumby proved the ‘four-cycle’ conjecture of Harary and Schwenk that every viable scheme of  $2n - 4$  phone calls contains a four-cycle: four dons,  $a, b, c$  and  $d$  such that all four of the calls  $ab, bc, cd$  and  $da$  are made at some stage during the process. Lebensold formulated and solved a generalization of the telephone problem *k-party conference calls*, i.e., calls in which  $k$  dons take part. (For simpler proofs and various extensions, see Bermond, and Kleitman and Shearer.) The extension of the  $2n - 4$  bound is that for  $n > k^2 \geq 4$  the minimal number of  $k$ -party conference calls that are needed to achieve that all dons know all items of gossip is  $2(n - k)(k - 1)$ .

B. Baker and R. Shostak, Gossips and telephones, *Discrete Math.* **2** (1972), 191–193.

J.-C. Bermond, Le problème des “ouvriers”, in *Problèmes Combinatoires et Théorie des Graphes* (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), Colloq. Internat. CNRS, 260, CNRS, Paris, (1978), pp. 31–34.

R.T. Bumby, A problem with telephones, *SIAM J. Algebraic Discrete Methods* **2** (1981), 13–18.

A. Hajnal, E.C. Milner and E. Szemerédi, A cure for the telephone disease, *Res. Paper No. 106*, Univ. Calgary, Alberta, Canada, January, 1971; also *Canad. Math. Bull.* **15** (1974), 447–450.

D.J. Kleitman and J.B. Shearer, Further gossip problems, *Discrete Math.* **30** (1980), 151–156.

K. Lebensold, Efficient communication by phone calls, *Studies in Appl. Math.* **52** (1973), 345–358.

R. Tijdeman, On a telephone problem, *Nieuw Arch. Wisk.* **3** (1971), 188–192.

## 49. Exact Covers: the de Bruijn–Erdős Theorem

Let  $\mathcal{L} = \{L_1, L_2, \dots, L_m\}$  be a collection of proper subsets of a set  $P = \{p_1, p_2, \dots, p_n\}$  of  $n$  elements (i.e.,  $|L_i| < n$  for every  $i$ ) such that any two elements of  $P$  are contained in exactly one set  $L_j$ . Then  $m \geq n$ .

Furthermore, equality holds if and only if either one of the  $L_j$  contains  $n - 1$  elements, and each of the other  $n - 1$  sets has two elements, or  $n$  is of the form  $n = k^2 + k + 1$ , each of the sets  $L_j$  consists of exactly  $k + 1$  elements, and the sets  $L_j$  are the lines of a projective plane with point set  $\{p_1, \dots, p_n\}$ .

*Proof.* As in the *Hint*, we write  $d_i$  for the number of lines through a point (the *degree* of  $p_i$ ), and  $D_j$  for the number of points contained in the line  $L_j$  (the *degree* of  $L_j$ ). As we may assume that every line contains at least two points, we have  $2 \leq d_i \leq n - 1$  and  $2 \leq D_j \leq n - 1$ . Also, counting the number of pairs  $(p_i, L_j)$  with  $p_i \in L_j$  in two different ways, we see that

$$\sum_{i=1}^n d_i = \sum_{j=1}^m D_j. \quad (14)$$

If a point  $p_i$  is not on a line  $L_j$ , then for every point of  $L_j$  there is a line through that point and  $p_i$ ; as all these lines are distinct,

$$d_i \geq D_j. \quad (15)$$

In words, every point not on  $L_j$  has degree at least  $D_j$ .

By appropriately renumbering the points and lines, we may assume that  $p_n$  is of minimal degree, the lines through it are  $L_1, L_2, \dots, L_s$ , and  $p_i$  is on  $L_i$  for each  $i$ ,  $1 \leq i \leq s$ . Note that each  $p_i$ ,  $1 \leq i \leq s$ , is on exactly one of the lines  $L_1, L_2, \dots, L_s$ , namely  $L_i$ . Consequently, by (15),

$$d_1 \geq D_2, \quad d_2 \geq D_3, \quad \dots, \quad d_s \geq D_1. \quad (16)$$

Furthermore, each of the lines  $L_{s+1}, L_{s+2}, \dots, L_m$  misses  $p_n$ , so has degree at most  $d_n$ . Combining this with (14), we find that  $m \geq n$ , completing the proof of the first assertion.

Let us turn to the proof of the second assertion. Suppose that  $m = n$ . Then we have equality in each of the relations in (16) and  $d_n = D_j$  for  $j > s$ , so the degree sequence  $(D_j)$  is only a permutation of  $(d_i)$ . Let us relabel the points and lines in such a way that  $d_1 = D_1 \geq d_2 = D_2 \geq \dots \geq d_n = D_n$ .

Suppose first that  $d_1 = D_1 > d_2 = D_2$ . As by (14) every point not on  $L_1$  has degree at least  $D_1$ , and  $d_2 < D_1$ , there is only one point not on  $L_1$ , as claimed.

To complete the proof, we have to take care of the case  $d_1 = D_1 = d_2 = D_2 = D$ , say. Since every pair of points is contained in exactly one line,

$$\binom{n}{2} = \sum_{j=1}^n \binom{D_j}{2} \leq n \binom{D}{2} \quad (17)$$

that is,

$$D(D-1) \geq n-1, \quad (18)$$

with equality if and only if each  $D_j$  is  $D$ . Note that if two lines containing  $D$  points each would not intersect then each of the  $D^2$  pairs of points meeting each of the lines would determine a distinct line, so we would have

$$D^2 + 2 \leq n,$$

contradicting (18). Hence any two lines containing  $D$  points intersect in exactly one point.

Let  $L_1$  and  $L_2$  intersect in the point  $p_i$ . Then  $d_i \geq (n-1)/(D-1)$ , and there are  $(D-1)^2$  lines meeting each of  $L_1$  and  $L_2$  that do not pass through  $p_i$ . Consequently,

$$\frac{n-1}{D-1} + (D-1)^2 \leq n. \quad (19)$$

Combining this with (18), we find that

$$D(D-1) \leq n-1.$$

Hence, we do have equality in (17), so all the lines have the same number of points, any two lines meet in exactly one point and any two points are contained in exactly one line, i.e., the lines are precisely the lines of a projective plane, as claimed.  $\square$

**Notes.** The result in the question is the well-known theorem that de Bruijn and Erdős proved in 1948. As it is with so many results in mathematics, the achievement was thinking of the assertion: having thought of the result, a proof is not far behind. In fact, almost every ‘head-on’ attack is likely to be successful: after all, we have relations (14), (15), (17), and the analogue of (19) for the lines  $L_1$  and  $L_2$  in our arsenal. The proof above is the elegant original proof of de Bruijn and Erdős; as they remarked, the first part, the assertion that  $m \geq n$ , was also proved by George Szekeres, but in a more complicated way.

For an extension of the de Bruijn–Erdős theorem, see the next question, Problem 50.

## 50. Constant Intersections: an Extension of the de Bruijn–Erdős Theorem

Let  $\mathcal{E} = \{E_1, E_2, \dots, E_n\}$  be a collection of subsets of  $[m] = \{1, \dots, m\}$  such that any two elements of  $\mathcal{E}$  intersect in exactly  $\lambda \geq 1$  elements. Then  $n \leq m$ .

*Proof.* First, note that if some  $E_i$  has only  $\lambda$  elements then every other  $E_j$  contains this  $E_i$  and so the sets  $E_j \setminus E_i$  are disjoint; consequently,  $n \leq m - \lambda + 1 \leq m$ . Therefore we may and shall assume that  $|E_i| > \lambda$  for every  $i$ .

Now, following the *Hint*, let us turn to the incidence matrix  $A$  and the adjacency matrix  $B$  (of the sets  $E_j$ ) of  $\mathcal{E}$ . Note that  $B = A^*A$ , where  $A^*$  is the transpose of  $A$ ; in particular, writing  $\text{rank} C$  for the rank of a matrix  $C$ ,

$$\text{rank} B \leq \text{rank} A \leq \min\{n, m\} \leq m.$$

On the other hand,  $B = \lambda J + D$ , where  $J$  is the  $n \times n$  matrix all of whose entries are 1, and  $D$  is the diagonal matrix  $\text{Diag}(d_1, \dots, d_n)$  with  $d_i = |E_i| - \lambda \geq 1$ . Since  $J$  is positive semidefinite and  $D$  is positive definite,  $B = \lambda J + D$  is positive definite; in particular, its rank is  $n$ . Consequently,  $n \leq m$ , completing our solution.

Let us note that the fact that  $B$  has rank  $n$  follows from first principles as well. Indeed, let  $b_1, \dots, b_n$  be the row vectors of  $B$ , and suppose  $S = \sum_{i=1}^n \mu_i b_i = 0$ . Setting  $\mu = \sum_{i=1}^m \mu_i$ , from  $S_i = 0$  we see that  $\mu_i = -\lambda \mu / d_i$  for every  $i$ . Substituting this into the expression for  $S$ , we find that  $\mu_i = 0$  for every  $i$ .  $\square$

**Notes.** The de Bruijn–Erdős inequality, presented as Problem 49, is, in fact, the dual of the case  $\lambda = 1$  of this result. Another natural way to present these assertions is in terms of bipartite graphs; in this setting the two forms (which are dual to each other) are naturally identified. Indeed, let  $G$  be an  $n$  by  $m$  bipartite graph, with vertex classes  $U$  and  $V$ ,  $|U| = n$  and  $|V| = m$ . Suppose that for every pair of vertices in  $U$  there are precisely  $\lambda$  vertices in  $V$  joined to both. Then  $n \leq m$ . Replacing the vertices in  $U$  by sets (lines) and those in  $V$  by elements (points), we get the de Bruijn–Erdős formulation in Problem 49, and taking them the other way, we get the formulation of the extension in this question.

The general inequality (i.e., for  $\lambda \geq 1$ ) is due to Majumdar (1953) and Isbell (1959); it extends Fisher's inequality (1940) as well. The beautiful proof above is due to Babai (1987); it is a modification of Bose's (1949) proof of Fisher's inequality. In the mid-1970s I heard this proof from Ernst Straus, who I believe never published it.

- L. Babai, On the nonuniform Fisher inequality, *Discrete Math.* **66** (1987), 303–307.
- R.C. Bose, Fisher's inequality for balanced incomplete block designs, *Ann. Math. Stat.* **20** (1949), 619–620.
- N.G. de Bruijn and P. Erdős, On a combinatorial problem, *Indagationes Math.* **10** (1948), 421–423.
- R.A. Fisher, An examination of the different possible solutions of a problem in incomplete blocks, *Ann. Eugenics (London)* **10** (1940), 52–75.
- J.R. Isbell, An inequality for incidence matrices, *Proc. Amer. Math. Soc.* **10** (1959), 216–218.
- K.M. Majumdar, On some theorems in combinatorics relating to incomplete block designs, *Ann. Math. Stat.* **24** (1953), 377–389.
- H.J. Ryser, An extension of a theorem of de Bruijn and Erdős on combinatorial design, *J. Algebra* **10** (1968), 246–261.
- D.R. Woodall, Square  $\lambda$ -linked designs, *Proc. Amer. Math. Soc.* **20** (1970), 669–687.

## 51. Bell Numbers

The  $n$ th Bell number  $B_n$  is given by the formula

$$B_n = \left\lceil \frac{1}{e} \sum_{k=1}^{2n} \frac{k^n}{n!} \right\rceil.$$

*Proof.* The formula we have to prove is due to Comtet; since

$$\sum_{k=2n+1}^{\infty} \frac{k^n}{k!} < e,$$

it suffices to prove *Dobiński's formula* that

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{n!}. \quad (20)$$

To justify formula (20), first we note the following recursion:

$$B_n = \sum_{k=1}^n \binom{n-1}{k-1} B_{n-k} = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k.$$

Indeed, this recursion holds since every partition of  $[n]$  can be obtained by first putting  $n$  into a part with  $k-1$  elements of the set  $[n-1]$ , and then partitioning the remaining  $n-k$  elements in any way.

Now, (20) holds for  $n \leq 1$ ; assuming that  $n \geq 2$  and (20) holds for smaller values of  $n$ , by the recursion formula we find that

$$\begin{aligned} B_n &= \sum_{k=1}^{n-1} \binom{n-1}{k} \frac{1}{e} \sum_{\ell=0}^{\infty} \frac{\ell^k}{k!} = \frac{1}{e} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{k=0}^{n-1} \binom{n-1}{k} \ell^k \\ &= \frac{1}{e} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (\ell+1)^{n-1} = \frac{1}{e} \sum_{\ell=0}^{\infty} \frac{\ell^n}{\ell!}, \end{aligned}$$

as required. □

**Notes.** Bell numbers occur frequently in combinatorics, algebra, probability theory and statistical physics; below we list some of the vast relevant literature. Here, let us note some relations concerning partition numbers related to the Bell numbers. Write  $b_{n,k}$  for the number of partitions of  $[n]$  into  $k$  parts. Thus,  $b_{n,1} = b_{n,n} = 1$ ; also,  $b_{n,2} = 2^{n-1} - 1$  since a partition of  $[n]$  into two parts can be identified with



a proper subset of  $[n - 1]$ . Clearly, for every  $n \geq 1$  we have  $b_{n,k} = 0$  if  $k > n$  or  $k \leq 0$ . By definition,  $B_n = \sum_{k=1}^n b_{n,k}$ .

A basic recursion can be obtained for  $b_{n,k}$  by answering the simple question: in a partition of  $[n]$ , is the part containing  $n$  a singleton? A moment's thought tells us that  $n$  is a singleton in  $b_{n-1,k-1}$  partitions, and is not a singleton in  $kb_{n-1,k}$  partitions. To see the latter, note that a partition of  $[n - 1]$  into  $k$  parts can be extended in  $k$  different ways to partitions of  $[n]$  by adding  $n$  to one of the  $k$  parts. Hence,

$$b_{n,k} = b_{n-1,k-1} + kb_{n-1,k}. \quad (21)$$

This recursion formula enables us to give other interpretations of the numbers  $b_{n,k}$ . For example, writing  $x_{(k)}$  for the  $k$ th *falling factorial* of  $x$ , i.e., setting  $x_{(k)} = x(x-1)(x-2)\cdots(x-k+1)$ , the numbers  $b_{n,k}$  are determined by the expansion

$$x^n = b_{n,1}x_{(1)} + b_{n,2}x_{(2)} + \cdots + b_{n,n}x_{(n)} = \sum_{k=1}^n b_{n,k}x_{(k)}. \quad (22)$$

It is trivial that (21) determines some real numbers  $b_{n,k}$ , and we can apply induction on  $n$  to show that these are our partition numbers  $b_{n,k}$ . Indeed, this is obvious for  $n = 1$ ; assuming that  $n \geq 2$  and the assertion holds for smaller values of  $n$ , we find that

$$\begin{aligned} x^n &= x \sum_{k=1}^{n-1} b_{n-1,k}x_{(k)} = \sum_{k=1}^{n-1} b_{n-1,k}x_{(k)}((x-k)+k) \\ &= \sum_{k=1}^{n-1} b_{n-1,k}x_{(k+1)} + \sum_{k=1}^{n-1} kb_{n-1,k}x_{(k)} = \sum_{k=1}^n (b_{n-1,k-1} + kb_{n-1,k})x_{(k)} \\ &= \sum_{k=1}^n b_{n,k}x_{(k)}, \end{aligned}$$

as claimed, with the last identity following from (21).

Another simple induction argument can be used to show that (21) implies that

$$b_{n,k} = \sum_{\ell=0}^{k-1} \frac{(-1)^\ell (k-\ell)^{n-1}}{\ell!(k-\ell-1)!}.$$

Consequently,

$$B_n = \sum_{k=1}^n b_{n,k} = \sum_{k=1}^n \sum_{\ell=0}^{k-1} \frac{(-1)^\ell (k-\ell)^{n-1}}{\ell!(k-\ell-1)!} = \sum_{m=1}^n \left( \sum_{\ell=1}^{n-m} \frac{(-1)^\ell}{\ell!} \right) \frac{m^{n-1}}{(m-1)!},$$

with the last identity following by setting  $m = k - \ell$ .

- Blasiak, P., Penson, K. A., and Solomon, A. I., Combinatorial coherent states via normal ordering of bosons, *Lett. Math. Phys.* **67** (2004), 13–23.
- Blasiak, P., Penson, K. A., and Solomon, A. I., Dobiński-type relations and the log-normal distribution, *J. Phys. A* **36** (2003), L273–L278.
- Comtet, L., *Advanced Combinatorics, The Art of Finite and Infinite Expansions*, Revised and enlarged edition, D. Reidel Publishing Co., (1974), xi + 343 pp.
- Klazar, M., Bell numbers, their relatives, and algebraic differential equations, *J. Combin. Theory Ser. A* **102** (2003), 63–87.
- Constantine, G. M., and Savits, T. H., A stochastic process interpretation of partition identities, *SIAM J. Discrete Math.* **7** (1994), 194–202.
- Pitman, J., Some probabilistic aspects of set partitions, *Amer. Math. Monthly* **104** (1997), 201–209.
- Rota, G.-C., The number of partitions of a set, *Amer. Math. Monthly* **71** (1964), 498–504.
- Wilf, H. S., *generatingfunctionology*, Second edition, Academic Press, Inc., (1994), x + 228 pp.

## 52. Circles Touching a Square

*At most six non-overlapping circular discs of radius 1 can touch a square of side-length 2.*

*Proof.* The main problem is that in an extremal arrangement, as in Figure 9 of the original problem, the six circles ‘rattle’: we have to show that the ‘elbow room’ is not enough to accommodate a seventh circle, no matter how we put down the first six.

Let  $Q$  be a (solid) square of side-length 2, and let  $C = ABCDEFGH$  be the curve formed by the points at distance exactly 1 from  $Q$ , as in Figure 58. Thus each of  $AB$ ,  $CD$ ,  $EF$  and  $GH$  is a quarter of a unit circle, and each of  $BC$ ,  $DE$ ,  $FG$  and  $HA$  is a segment of length 2.

The centre of a circular disc touching  $Q$  lies on  $C$ , so our task is to show that every heptagon inscribed in  $C$  has a side of length *strictly less than* 2, i.e., no matter how we place seven points on  $C$ , some two *neighbouring* points are at distance strictly less than 2.

The example of six circles mentioned in Problem 52 gives a hexagon in which every side has length at least 2, namely the hexagon  $XBCYFG$ , where  $X$  is the midpoint of  $HA$  and  $Y$  the midpoint of  $DE$  (see Figure 58). A half-hearted attempt at inscribing a suitable heptagon is shown in Figure 59.

The perimeter length of  $C$  is  $8 + 2\pi = 14.283\dots$ , so this does not rule out the possibility of inscribing a heptagon into  $C$  with  $0.283\dots$  to spare. Hence our task

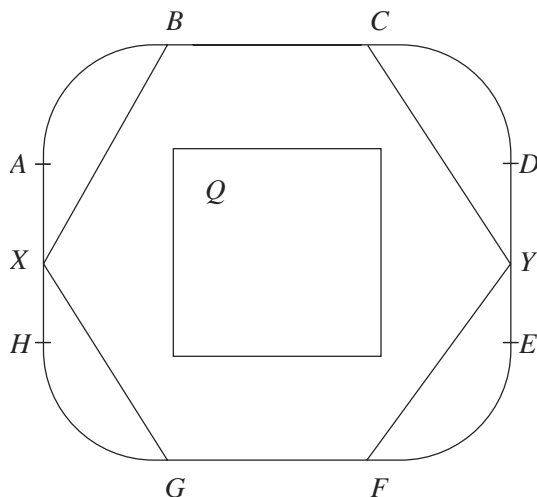


Fig. 58. The curve  $C$  around the square  $Q$ , and the hexagon  $XBCYFG$ .

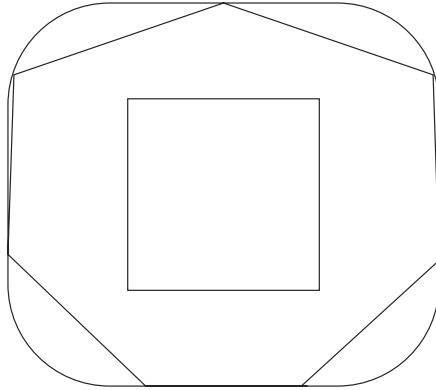


Fig. 59. A heptagon in  $C$ , with six of its sides having length 2.

is to show that the four quarter circles force us to lose more than  $0.283 \dots$ . This is easily done by elementary (although somewhat tedious) calculus: when we get around two neighbouring corners then we lose at least  $0.142$ .  $\square$

**Notes.** The problem is probably folklore, but it seems that it first appeared as a problem posed by Bill Sands in the May 1998 issue of *Mathematics and Informatics*.

### 53. Gambling

In a game of  $n$  gamblers, the  $i$ th gambler starts the game with  $a_i$  dollars. In each round, two gamblers selected at random make a fair bet, and the winner gets a dollar from the loser. A gambler losing all his money leaves the game. The game continues as long as possible, i.e., until one of the gamblers has all the money. Then:

- (i) the expected number of rounds is  $\sum_{i < j} a_i a_j$ ;
- (ii) the probability that the  $i$ th gambler ends up with all the money is  $a_i / \sum_{j=1}^n a_j$ ;
- (iii) for  $n = 3$ , the expected number of rounds played by all three players is  $3a_1 a_2 a_3 / (a_1 + a_2 + a_3)$ .

*Proof.* In each case, the problem is only to guess the answer, so by giving away the answers, the problem loses its sting. Indeed, it is easy to write down a recurrence relation and a boundary condition. In addition to checking that our guess satisfies these relations, all we have to note is that the solution is unique. The latter is shown as one proves the uniqueness of a harmonic function with given boundary conditions: the difference of two solutions has maximum 0 and minimum 0.

Let us say that the game is in position  $(a_1, \dots, a_n)$  if, for  $i = 1, \dots, n$ , the  $i$ th player has  $a_i$  dollars. In one round, the position can change to any one of the neighbouring positions obtained from  $(a_1, \dots, a_n)$  by decreasing an  $a_i$  by one and increasing an  $a_j$  by one, provided both  $a_i$  and  $a_j$  are positive. Let us write  $N(a_1, \dots, a_n)$  for the set of positions neighbouring  $(a_1, \dots, a_n)$ . Thus, if  $k$  of the  $a_i$  are positive then  $|N(a_1, \dots, a_n)| = k(k-1)$ . The set of positions we can reach if we start from  $(a_1, \dots, a_n)$ , with each  $a_i$  positive, is  $V(a_1, \dots, a_n) = \{(x_1, \dots, x_n) : \sum x_i = a \text{ and } x_i \geq 0 \text{ for every } i, \text{ with equality whenever } a_i = 0\}$ , where  $a = \sum_{i=1}^n a_i$  is the total amount owned by the players. Starting from position  $(a_1, \dots, a_n)$ , the entire game is played on the set  $V(a_1, \dots, a_n)$ , so we are interested in various functions defined on this set.

(i) Let  $r(a_1, \dots, a_n)$  be the expected number of rounds till a player has all the money. The value  $r(a_1, \dots, a_n)$  is one more than the average of  $r$  on the set  $N(a_1, \dots, a_n)$  of the  $k(k-1)$  neighbours of  $(a_1, \dots, a_n)$ , where  $k$  is the number of strictly positive  $a_i$ .

Our boundary conditions are the following: if all but the  $i$ th coordinate is 0 then  $r = 0$ , i.e.,  $r(0, \dots, 0, a_i, 0, \dots, 0) = 0$ . Clearly, the function

$$r(a_1, \dots, a_n) = \sum_{i < j} a_i a_j \quad (23)$$

satisfies the recurrence relations and the boundary conditions. We claim that this solution is unique. Indeed, if  $r'$  is another solution and  $s = r - r'$  then  $s(a_1, \dots, a_n)$

is the average of its  $k(k-1)$  values on the neighbourhood  $N(a_1, \dots, a_n)$  of  $(a_1, \dots, a_n)$ . Hence if the maximum  $m$  of  $s$  is attained at  $(a_1, \dots, a_n)$  then  $s$  is identically  $m$  on  $N(a_1, \dots, a_n)$ . Consequently,  $s$  is identically  $m$  on the entire set  $V(a_1, \dots, a_n)$  that can be reached from  $(a_1, \dots, a_n)$ . Since this set meets the boundary,  $m = 0$ , i.e.,  $r \leq r'$ . Furthermore, since  $r$  and  $r'$  can be interchanged, we have  $r' \leq r$  as well, so  $r = r'$ , as claimed. Therefore (23) is indeed the expected number of rounds.

(ii) Let  $p_i(a_1, \dots, a_n)$  be the probability that the  $i$ th gambler ends up with all the money. Clearly,  $p_i(a_1, \dots, a_n)$  is the average of the values of  $p_i$  on the neighbourhood of  $(a_1, \dots, a_n)$ , and the boundary conditions are  $p_i(a_1, \dots, a_n) = 0$  whenever  $a_i = 0$  and  $p_i(a_1, \dots, a_n) = 1$  if  $a_i = a$  (and so  $a_j = 0$  for all  $j \neq i$ ). It is trivial to check that

$$p_i(a_1, \dots, a_n) = \frac{a_i}{\sum_{j=1}^n a_j} = \frac{a_i}{a}$$

is the unique solution, so the probability in question is indeed  $a_i \sum_{j=1}^n a_j$ .

(iii) As here  $n = 3$ , we write  $a, b, c$  for the amounts held by the players and  $f(a, b, c)$  for the expected number of rounds till the first of the players (whichever he is) goes bankrupt. It is easily seen that  $f(a, b, c) = 3abc/(a+b+c)$ . Indeed,  $f$  satisfies the same recurrence relation as the round function  $r$  in (i): if each of  $a, b$  and  $c$  is at least 1 then  $f(a, b, c)$  is 1 more than the average of  $f$  on the six neighbours of  $(a, b, c)$ . However, the boundary conditions are different: if  $abc = 0$  then  $f(a, b, c) = 0$ . It is easily seen that  $f(a, b, c) = 3abc/(a+b+c)$  satisfies these conditions; uniqueness follows as before.  $\square$

**Notes.** This is one of the many questions I used when interviewing after the Scholarship Exam in Cambridge in the early 1970s. The second part was usually guessed, and uniqueness was eventually proved, but the other parts tended to cause difficulties.

## 54. Complex Sequences

*Let  $z_1, z_2, \dots$  be a sequence of complex numbers converging to 0. Then there are signs  $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1, \dots$  such that the series  $\sum_{i=1}^{\infty} \varepsilon_i z_i$  is convergent.*

*Proof.* The key observation is that to every sequence of complex numbers of modulus at most 1, one can assign signs such that all the partial sums of this ‘signed’ series have modulus at most  $C$  for some constant  $C$ . A moment’s thought tells us that this assertion is, in fact, equivalent to the full result. As it happens, it is easy to prove the assertion with the optimal value of  $C$ , as we are about to show.

**Claim.** *For every sequence  $z_1, z_2, \dots, z_N$  of complex numbers of modulus at most 1, there is a sequence of signs,  $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1, \dots, \varepsilon_N = \pm 1$ , such that  $|\sum_{i=1}^n \varepsilon_i z_i| \leq \sqrt{3}$  for every  $n, 1 \leq n \leq N$ .*

*Proof of the Claim.* We apply induction on  $N$ . As for  $N = 1$  there is nothing to prove, and for  $N = 2$  the assertion is obvious since at least one of  $|z_1 + z_2|$  and  $|z_1 - z_2|$  is at most  $\sqrt{2}$ , we turn to the proof of the induction step. Thus, suppose  $N \geq 3$  and the Claim holds for a sequence of  $N - 1$  terms. The idea is very simple: we replace some two of the first three terms,  $z_i$  and  $z_j$ , say, by  $z_i \pm z_j$  of modulus at most 1 and apply the induction hypothesis to the new sequence  $(w_i)$  of length  $N - 1$ .

(i) First, suppose that  $\min_{\pm} |z_1 \pm z_2| \leq 1$ ; say,  $|z_1 + z_2| \leq 1$ . Set  $w_2 = z_1 + z_2$  and  $w_i = z_i$  for  $3 \leq i \leq N$ . By the induction hypothesis, there are signs  $\delta_2 = 1, \delta_3 = \pm 1, \dots, \delta_N = \pm 1$  such that  $|\sum_{i=2}^n \delta_i w_i| \leq \sqrt{3}$  for every  $n, 2 \leq n \leq N$ . Then the signs  $\varepsilon_1 = \varepsilon_2 = 1$  and  $\varepsilon_i = \delta_i$  for  $3 \leq i \leq N$  will do for the sequence  $(z_i)_{i=1}^N$ .

(ii) Next, suppose that  $\min_{\pm} |z_1 \pm z_2| > 1$  (i.e., case (i) does not hold) and  $\min_{\pm} |z_1 \pm z_3| \leq 1$ ; say,  $|z_1 + z_3| \leq 1$ . Let  $\delta_2 = 1, \delta_3 = \pm 1, \dots, \delta_N = \pm 1$  be a suitable sequence of signs for the sequence  $w_2 = z_1 + z_3, w_3 = z_2$  and  $w_i = z_i$  for  $4 \leq i \leq N$ . Then  $\varepsilon_1 = 1, \varepsilon_2 = \delta_3, \varepsilon_3 = 1$ , and  $\varepsilon_i = \delta_i$  for  $4 \leq i \leq N$  will do for  $(z_i)_{i=1}^N$ .

(iii) This is the final case. Suppose that  $\min_{\pm} |z_1 \pm z_2| > 1$  and  $\min_{\pm} |z_1 \pm z_3| > 1$ , i.e., neither of the previous cases holds. Then the angle between the lines of  $z_1$  and  $z_2$  is greater than  $\pi/3$ , and so is the angle between the lines  $z_1$  and  $z_3$ . Consequently the angle between the lines of  $z_2$  and  $z_3$  is less than  $\pi/3$ , so  $\min_{\pm} |z_2 \pm z_3| < 1$ ; say,  $|z_2 + z_3| < 1$ . Let  $\delta_2 = 1, \delta_3, \dots, \delta_N$  be a suitable sequence of signs for  $w_2 = z_1, w_3 = z_2 + z_3, w_i = z_i$  for  $4 \leq i \leq N$ . Then  $\varepsilon_1 = 1, \varepsilon_2 = \delta_3, \varepsilon_3 = \delta_3$  and  $\varepsilon_i = \delta_i$  for  $4 \leq i \leq N$  will do for  $(z_i)_{i=1}^N$ , completing the proof.  $\square$

**Notes.** This was one of my favourite problems in Cambridge for the first year analysis examples sheet for the enthusiasts: as far as I knew, I had thought of the problem. It was quite a shock when, many years later, during one of my visits to Israel, I discovered that Dvoretzky and Hanani had proved considerably more about twenty years earlier, and the theme was continued by Calabi and Dvoretzky.

E. Calabi and A. Dvoretzky, Convergence- and sum-factors for series of complex numbers, *Trans. Amer. Math. Soc* **70** (1951), 177–194.

A. Dvoretzky and Ch. Hanani, Sur les changements des signes des termes d’une série a termes complexes, *C. R. Acad. Sci. Paris* **225** (1947), 516–518.



## 55. Partitions of Integers

(i) The number of partitions of  $n$  into  $k$  distinct summands in which the maximum of the summands is  $m$  is equal to the number of partitions of  $n$  into  $m$  summands in which the summands are  $k, k-1, \dots, 1$ , each with multiplicity at least 1.

(ii) The number of partitions of an integer into distinct summands equals the number of its partitions into odd summands.

*Proof.* With a slight abuse of notation, we shall write a partition of  $n$  as a vector  $\mathbf{x} = (x_1, \dots, x_k)$ , although, strictly speaking a partition is an unordered multiset of natural numbers summing to  $n$ . To make the representation unique, it is customary to write the components in decreasing order; for example,  $(9, 7, 7, 7, 3, 3, 1)$  is a partition of 37 into seven odd components.

(i) Represent a partition  $\mathbf{x} = (x_1, \dots, x_k)$ ,  $x_1 \geq x_2 \geq \dots \geq x_k$ , by its *Young diagram* consisting of  $x_1$  cells in the first row,  $x_2$  in the second, etc., as in Figure 60. Thus a Young diagram consists of  $n$  cells of an  $n \times n$  chess board placed in the upper left corner of the board such that whenever a cell is in the diagram, so are the cells above it and to the left of it. By construction, the maximum of the summands in a partition is the number of columns of its Young diagram, and the number of summands is the number of rows. Also, two rows have the same number of cells if and only if there is a decrease of at least two in the (decreasing) sequence of columns (properly, in the sequence of the numbers of cells in the columns).

Note that the reflection of a Young diagram in its main diagonal is again a Young diagram: this reflection gives us the required one-to-one correspondence between partitions into  $k$  distinct summands, with maximum  $m$ , and partitions with  $k$  summands, in which the summands are  $k, k-1, \dots, 1$  with various multiplicities (each with multiplicity at least 1).

(ii) Let  $\mathcal{D}(n)$  be the set of partitions of  $n$  into distinct summands, and  $\mathcal{O}(n)$  the set of partitions into odd summands. We shall establish a very simple one-to-one

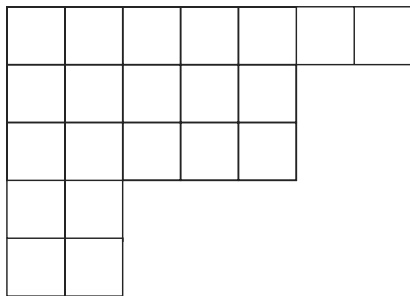


Fig. 60. The Young diagram of the partition  $21 = 7 + 5 + 5 + 2 + 2$ , i.e., of the vector  $(7, 5, 5, 2, 2)$ .

correspondence between  $\mathcal{D}(n)$  and  $\mathcal{O}(n)$ . Given a partition  $\mathbf{x} = (x_1, \dots, x_k) \in \mathcal{D}(n)$ , write each  $x_i$  as  $2^{\mu_i} y_i$ , where  $y_i$  is odd, and let  $\mathbf{y} = \varphi(\mathbf{x})$  be the partition obtained from  $\mathbf{x}$  by replacing  $x_i$  by  $y_i$  repeated  $2^{\mu_i}$  times. For example if  $\mathbf{x} = (6, 5, 4)$  then  $\varphi(\mathbf{x}) = (5, 3, 3, 1, 1, 1, 1)$ . Clearly,  $\varphi(\mathbf{x}) \in \mathcal{O}(n)$ .

To construct the inverse of  $\varphi$ , given a partition  $\mathbf{y} \in \mathcal{O}(n)$ , let a component  $y_i$  of  $\mathbf{y}$  be repeated  $m_i$  times. Write  $m_i$  as a sum of distinct powers of 2:

$$m_i = \sum_{j=1}^{\ell} 2^{\mu_j}, \quad \text{where } \mu_1 > \dots > \mu_{\ell},$$

and replace the block  $(y_i, \dots, y_i)$  of  $\mathbf{y}$  by the block  $(2^{\mu_1} y_i, 2^{\mu_2} y_i, \dots, 2^{\mu_{\ell}} y_i)$  to obtain a partition  $\psi(\mathbf{y})$  of  $n$  into distinct summands. For example,  $\mathbf{y} = (7, 5, 5, 3, 3, 3, 1, 1)$  is mapped into  $\psi(\mathbf{y}) = (10, 7, 6, 3, 2)$ . Clearly,  $\psi$  is the inverse of  $\varphi$ , so  $\varphi$  and  $\psi$  give us a one-to-one correspondence between  $\mathcal{D}(n)$  and  $\mathcal{O}(n)$ .  $\square$

**Notes.** Although the study of partitions goes back to Leibniz and Euler, it became an established area of mathematics in the early 20th century when a number of Cambridge mathematicians (MacMahon, Rogers, Hardy, Ramanujan and others) took an active interest in it. The results in the question are just about the simplest non-trivial results concerning partitions: they can be found in most books on the subject. In particular, (i) is rather trivial whether we use Young diagrams or not: given a partition  $\mathbf{x} = (x_1, \dots, x_k)$ , let  $\mathbf{x}^* = (y_1, \dots, y_{\ell})$ ,  $y_1 = k$ ,  $\ell = x_1$ , be the partition in which  $y_j$  is the number of  $x_i$  that are at least  $j$ . Then the ‘duality’  $\mathbf{x} \rightarrow \mathbf{x}^*$  establishes a one-to-one correspondence between partitions in which the maximum of the summands is  $k$  and partitions with exactly  $k$  summands. This duality also gives a one-to-one correspondence between partitions with distinct summands and partitions without a jump in the sequence of components (with repeats allowed, of course).

The clever (although rather natural) correspondence used in (ii) was given by Glaisher.

Euler was the first to notice that generating functions provide a systematic way of studying partitions. Writing  $p(n)$  for the number of partitions of  $n$ , and taking  $p(0) = 1$ , he observed that

$$\begin{aligned} & p(0) + p(1)x + p(2)x^2 + \dots \\ &= (1 + x + x^2 + x^3 + \dots)(1 + x^2 + x^4 + x^6 + \dots)(1 + x^3 + x^6 + x^9 + \dots) \dots, \end{aligned}$$

and so

$$p(0) + p(1)x + p(2)x^2 + \dots = \frac{1}{(1-x)(1-x^2)(1-x^3)\dots}.$$

Furthermore, Euler proved the beautiful and surprising *pentagonal theorem* that

$$\prod_{k=1}^{\infty} (1 - x^k) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots = \sum_{q=-\infty}^{\infty} (-1)^q x^{\frac{3q^2+q}{2}}.$$

This implies that

$$\sum_{n=0}^{\infty} p(n)x^n \sum_{q=-\infty}^{\infty} (-1)^q x^{\frac{3q^2+q}{2}} = 1,$$

and hence the recurrence formula

$$\begin{aligned} p(n) &= p(n-1) + p(n-2) - p(n-5) - p(n-7) + \dots \\ &= \sum_{q=1}^{\infty} (-1)^{q+1} \left\{ p\left(n - \frac{3q^2-q}{2}\right) + p\left(n - \frac{3q^2+q}{2}\right) \right\}, \end{aligned}$$

with  $p(n)$  taken to be 0 whenever  $n < 0$ . Using this recurrence formula MacMahon computed the first 200 values of  $p(n)$ ; later these values were found rather useful by Hardy and Ramanujan when they gave an amazing asymptotic formula for  $p(n)$ .

Let us note that part (ii) of our question is easily proved with the aid of generating functions as well. To this end, let  $d(n) = |\mathcal{D}(n)|$  be the number of partitions of  $n$  into distinct summands, and let  $o(n) = |\mathcal{O}(n)|$  be the number of partitions into odd summands, with the convention that  $d(0) = o(0) = 1$ . Then the generating functions are

$$\sum_{k=0}^{\infty} d(k)x^k = (1+x)(1+x^2)(1+x^3)\dots = \prod_{k=1}^{\infty} (1+x^k),$$

and

$$\begin{aligned} \sum_{k=0}^{\infty} o(k)x^k &= (1+x+x^2+\dots)(1+x^3+x^6+\dots)(1+x^5+x^{10}+\dots)\dots \\ &= 1 / \prod_{k=1}^{\infty} (1-x^{2k-1}). \end{aligned}$$

Since

$$1+x^k = \frac{1-x^{2k}}{1-x^k},$$

we find that the two generating functions are indeed equal:

$$\sum_{k=0}^{\infty} d(k)x^k = \prod_{k=1}^{\infty} \frac{1-x^{2k}}{1-x^k} = \frac{\prod_{k=1}^{\infty} (1-x^{2k})}{\prod_{k=1}^{\infty} (1-x^k)} = \frac{1}{\prod_{k=1}^{\infty} (1-x^{2k-1})} = \sum_{k=0}^{\infty} o(k)x^k.$$

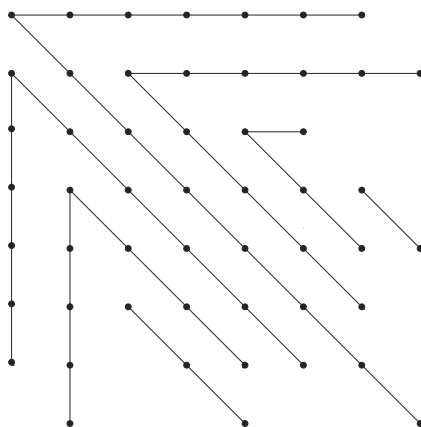


Fig. 61. Sylvester's representation of the partition  $(13, 13, 7, 7, 7, 3, 1, 1)$  of 52. By grouping the points as indicated in the figure, we get the partition  $(14, 11, 10, 8, 4, 3, 2)$ .

In conclusion, let us give another proof of part (ii): this time we shall use the correspondence defined by Sylvester. Represent a partition  $\mathbf{x} = (x_1, \dots, x_k) \in \mathcal{O}(n)$  by an array of points of an  $n \times n$  grid by placing  $x_1, x_2, \dots$  points into L-shaped regions symmetrically about the main diagonal, as in Figure 61. Thus, we use the first  $(x_1 + 1)/2$  points of the first row and the first  $(x_1 + 1)/2$  points of the first column to represent  $x_1$ , then delete the first row and column, and repeat the construction: we place  $(x_2 + 1)/2$  points into the first row and first column of the remaining  $(n - 1) \times (n - 1)$  grid, etc. Now comes Sylvester's beautiful idea: group the points as in Figure 61 to obtain a partition  $\sigma(\mathbf{x})$  of  $n$  into distinct summands. It is easily checked that this map  $\sigma : \mathcal{O}(n) \rightarrow \mathcal{D}(n)$  is a one-to-one correspondence.

G.E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, Vol. 2, Addison-Wesley, (1976), xiv + 255 pp. Reprinted by Cambridge University Press, 1985, 1998.

P. Diaconis, *Group Representations in Probability and Statistics*, Institute of Mathematical Statistics Lecture Notes—Monograph Series, **11**, Institute of Mathematical Statistics, (1988), vi + 198 pp.

G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Fifth edition, The Clarendon Press, Oxford University Press, (1979), xvi + 426 pp.

W. Fulton, *Young Tableaux, with Applications to Representation Theory and Geometry*, London Math. Soc. Student Texts, **35**, Cambridge University Press, (1997), x + 260 pp.

## 56. Emptying Glasses

*Let three large glasses (each of volume at least  $a + b + c$ ) contain  $a$ ,  $b$  and  $c$  integer ounces of water. In a legal move we double the contents of one glass from another.*

- (i) *There is a sequence of legal moves producing an empty glass.*
- (ii) *Let  $d$  be the greatest common divisor of  $a$ ,  $b$  and  $c$ . Then any two of the glasses can be emptied through legal moves if and only if  $(a + b + c)/d$  is a power of 2.*

*Proof.* (i) We may assume that  $0 < a \leq b \leq c$ , with  $a$  ounces of water in glass  $A$ ,  $b$  ounces in  $B$ , and  $c$  ounces in  $C$ . It suffices to show that after a sequence of doubling (legal) moves we can achieve that one of the glasses contains strictly less than  $a$  ounces of water.

Let  $r$  be the remainder of  $b$  when divided by  $a$ ; thus  $b = qa + r$  for integers  $q$  and  $r$  with  $0 \leq r < a$ . To prove the result, we shall show that  $qa$  ounces can be poured from  $B$  into  $A$ , leaving  $r < a$  ounces in glass  $B$ .

If  $q = 2^k - 1$  for some integer  $k$ , this is easily done: first we double the contents of  $A$  from  $B$ , then we double that, and so on; after  $k$  moves we have poured  $a + 2a + \dots + 2^{k-1}a = (2^k - 1)a = qa$  ounces of water into  $A$ , leaving  $r$  in  $B$ , as required.

What shall we do if  $q$  is not of this form? We shall still keep doubling the contents of  $A$ , but we shall invoke the help of the third glass: at each step, we pour into  $A$  either from  $B$  or from  $C$ . To be precise, take the binary representation of  $q$ :  $q = \sum_{i=0}^n q_i 2^i$ , where  $q_n = 1$  and each  $q_i$  is either 0 or 1. Now, make  $n + 1$  legal moves as follows: if  $q_i = 0$ , then in the  $(i + 1)$ st move pour from  $C$  into  $A$ , and if  $q_i = 1$  then double the contents of  $A$  from  $B$ . Note that, before the  $(i + 1)$ st move, glass  $A$  contains precisely  $2^i a$  ounces of water, so the total amount we pour from  $B$  is exactly  $\sum_{i=0}^n q_i 2^i a = qa$ ; in particular, at every step there is enough water in  $B$  to double the contents of  $A$ .

To conclude the proof, let us note that the moves involving  $C$  (pouring from  $C$  into  $A$ ) can also be performed. Indeed, if in the  $(i + 1)$ st move we pour from  $C$  (since  $q_i = 0$ ) then before that move we have poured  $(2^i - 1)a$  amount into  $A$ , so that  $A$  contains  $2^i a \leq 2^{n-1} a$  ounces, and  $C$  has at least

$$c - (2^i - 1)a \geq b - (2^i - 1)a \geq (2^n - 2^i + 1)a \geq (2^n - 2^{n-1} + 1)a = (2^{n-1} + 1)a$$

ounces.

- (ii) We may clearly assume that the numbers  $a, b, c$  are relatively prime, i.e.,  $d = 1$ . Now, if an odd prime  $p$  does not divide all three of  $a, b$  and  $c$ , and  $(a', b', c')$  is obtained from  $(a, b, c)$  by a legal move, then  $p$  does not divide all three of  $a'$ ,

$b'$  and  $c'$  either. Hence, if the contents of all three glasses can be poured into one, then the number of ounces in that glass,  $a + b + c$ , is not a multiple of  $p$ .

Conversely, if  $a + b + c = 2^k$  then we *can* empty two glasses. Let us see this by induction on  $k$ . For  $k = 0$  this is clear. To prove the induction step, first empty one glass – this can be done by part (i). At this stage we have  $a'$  in one glass and  $b' \geq a'$  in another, with  $a' + b' = 2^k \geq 2$ . Doubling  $a'$ , we get to the state  $(2a', b' - a')$ , with both  $2a'$  and  $b' - a'$  even. Now, this state is equivalent to  $(a', (b' - a')/2)$ ; as the new sum is  $2^{k-1}$ , by the induction hypothesis we can empty one of the glasses.  $\square$

**Notes.** I first heard the first part of this problem from Christian Borgs of Microsoft Research: at the annual *Tech Fest* at Microsoft, the *Theory Group* was set that problem by one of the visitors.

## 57. Distances in Planar Sets

Let  $P_1, \dots, P_n$  be  $n$  points in the plane.

- (i) There is a point  $P_i$  such that the distances  $P_i P_j$ ,  $i \neq j$ , take at least  $(n - 3/4)^{1/2} - 1/2$  different values.
- (ii) The maximal distance occurs at most  $n$  times.
- (iii) The minimal distance occurs at most  $3n - 6$  times.

*Proof.* (i) Write  $m$  for the maximal number of distinct distances from one of the points, and let  $P_\ell$  be a left-most point. Denote by  $s$  the number of distinct distances from  $P_\ell$ , and let  $t$  be the maximal number of points  $P_i$  at the same distance from  $P_\ell$ . Then

$$st \geq n - 1 \quad \text{and} \quad m \geq \max\{s, t - 1\},$$

since if we have  $t$  points on a semicircle then no two points are at the same distance from an extreme point. Therefore  $m(m + 1) \geq n - 1$ , implying that  $m \geq (n - 3/4)^{1/2} - 1/2$ .

(ii) Note that if  $P_i P_j$  and  $P_k P_\ell$  are maximal distances then the segments  $P_i P_j$  and  $P_k P_\ell$  intersect. Also, if  $P_i, P_j$  and  $P_k$  are at maximal distance  $D$  from a point  $P_\ell$  then for one of these three points (the one in the ‘middle’), the point  $P_\ell$  is the only point at distance  $D$ . This implies that  $D$  occurs at most  $n$  times. Figure 62 shows that for  $n \geq 3$  the maximal distance can occur  $n$  times.

(iii) Join two points by a straight-line segment if their distance is the minimal distance  $d$ . This gives a straight-line planar drawing of a graph, so the number of

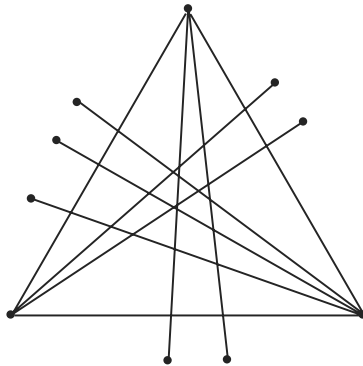


Fig. 62. An arrangement of ten points, with the maximal distance occurring ten times.

edges is at most  $3n - 6$ . A portion of the triangular lattice shows that the minimal distance may occur  $3n - O(n^{1/2})$  times.  $\square$

**Notes.** The results are from the very influential brief note Erdős wrote in 1946, posing several still unsolved problems about distances in planar sets. As he remarks there, by a more complicated argument one can show that the minimal distance occurs at most  $3n - cn^{1/2}$  times, where  $c > 0$  is an absolute constant.

P. Erdős, On sets of distances of  $n$  points, *Amer. Math. Monthly* **53** (1946), 248–250.



## 58. Monic Polynomials

For  $n \geq 1$ , let  $p_n(x)$  be a monic polynomial of degree  $n$  with real coefficients. Then  $|p_n(x)| \geq 2$  for some  $x$  with  $-2 \leq x \leq 2$ . Furthermore, there need not be a point  $x$  with  $-2 \leq x \leq 2$  and  $|p_n(x)| > 2$ .

*Proof.* For  $n \geq 1$  the function

$$T_n(x) = 2 \cos(n \cos^{-1}(x/2))$$

is the  $n$ th Chebyshev polynomial (of the first kind). This polynomial has several variants; perhaps the most common form is  $\cos(n \cos^{-1} x)$ : for  $n \geq 1$  this is a polynomial of degree  $n$  with leading coefficient  $2^{n-1}$ . In the context of our question, this form would be less convenient.

We claim that  $T_n$  has the following basic properties:

- (i) it is a monic polynomial of degree  $n$ ;
- (ii)  $|T_n(x)| \leq 2$  for  $|x| \leq 2$ ;
- (iii) there are  $n + 1$  points  $x_0 = -2 < x_1 < \cdots < x_n = 2$  such that  $T_n(x_k) = (-1)^{n+k} 2$ .

Indeed, for  $|x| \leq 2$ , let  $0 \leq \theta \leq \pi$  be given by  $\cos \theta = x/2$ . Then  $x = 2 \cos \theta = e^{i\theta} + e^{-i\theta}$ . By induction on  $n$ , it is easy to see that a trigonometric polynomial  $h(\theta) = \sum_{k=0}^n a_k (e^{ik\theta} + e^{-ik\theta})$  in which each  $a_k$  is real and  $a_n = 1$  is a monic real polynomial of degree  $n$  in  $x = e^{i\theta} + e^{-i\theta}$ . Indeed,  $p_n(x) - x^n = \sum_{k=0}^{n-1} a'_k (e^{ik\theta} + e^{-ik\theta})$ , where  $a'_{n-2\ell} = a_{n-2\ell} - \binom{n}{\ell}$  for  $1 \leq \ell \leq n/2$ , and  $a'_k = a_k$  if  $n - k$  is odd. Hence (i) holds.

The form  $T_n(x) = 2 \cos n\theta$  shows that (ii) holds as well. Finally, to see (iii), for  $k = 0, \dots, n$ , set  $x_k = 2 \cos \theta_k$ , where  $\theta_k = \pi - k\pi/n$ .

Turning to the question itself, note first that we have already answered the second part:  $p_n(x) = T_n(x)$  is a monic real polynomial of degree  $n$  whose maximum modulus in  $[-2, 2]$  is 2.

The first part is almost as easily answered. Suppose for a contradiction that  $p_n(x)$  is a monic real polynomial of degree  $n \geq 1$  with  $|p_n(x)| < 2$  whenever  $-2 \leq x \leq 2$ . Let  $x_0 = -2 < x_1 < \cdots < x_n = 2$  be as above, so that on each interval  $[x_k, x_{k+1}]$  the polynomial  $T_n(x)$  goes from  $(-1)^{n+k} 2$  to  $(-1)^{n+k+1} 2$ . As  $p_n(x)$  and  $T_n(x)$  are continuous, there are points  $x_0 = -2 < y_1 < x_1 < y_2 < x_2 < \cdots < y_n < x_n = 2$  such that  $T_n(y_k) = p_n(y_k)$  for  $k = 1, 2, \dots, n$ . Since  $T_n$  and  $p_n$  are monic polynomials, this implies that  $p_n(x) - T_n(x)$  is a polynomial of degree at most  $n - 1$ , with at least  $n$  zeros. Therefore  $p_n(x) = T_n(x)$ . In particular,  $p_n(2) = 2$ , contradicting our assumption on  $p_n$ , and completing our proof.  $\square$

**Notes.** Over forty years ago Paul Erdős gave a graduate course on trigonometric polynomials and approximation theory in Budapest; this question was one of many exercises he presented in the course.

The standard notation  $T_n(x)$  for a Chebyshev polynomial seems strange at first; however, the letter  $T$  stands for *Tschebyscheff*, the German transliteration of the name of the great Russian mathematician.



## 59. Odd Clubs

*Suppose that in a town with population  $n$ , every club has an odd number of members and any two clubs have an even number of members in common. Then there are at most  $n$  clubs.*

*Proof.* Clearly, we may have  $n$  clubs; for example, each inhabitant of Oddtown may form a (single) club without any other members.

To prove that we cannot have more clubs, let us reformulate the question using mathematical terms. Let  $A_1, \dots, A_m$  be odd subsets of  $[n] = \{1, 2, \dots, n\}$  such that each intersection  $A_i \cap A_j$  is even. (Thus  $|A_i|$  is odd for every  $i$  and  $|A_i \cap A_j|$  is even whenever  $i \neq j$ .) At most how large is  $m$ ?

To show that  $m$  is at most  $n$ , represent each subset  $X$  of  $[n]$  by its characteristic vector  $\mathbf{x} = \chi_X = (x_1, \dots, x_n)$ :

$$x_i = \begin{cases} 0 & \text{if } i \notin X, \\ 1 & \text{if } i \in X. \end{cases}$$

Let  $\mathbf{a}_1, \dots, \mathbf{a}_m$  be the characteristic vectors of the sets  $A_1, \dots, A_m$ , and let us consider these vectors as elements of the  $n$ -dimensional vector space  $V = \mathbb{Z}_2^n$  over the field  $\mathbb{Z}_2$  with two elements. This vector space is endowed with an inner product: if  $\mathbf{x} = (x_1, x_2, \dots, x_n), \mathbf{y} = (y_1, y_2, \dots, y_n) \in V$  then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i.$$

With this inner product, the vectors  $\mathbf{a}_1, \dots, \mathbf{a}_m$  are pairwise orthogonal (i.e., the inner product of any two is 0), and each  $\mathbf{a}_i$  has an odd number of 1-coordinates. We claim that this implies that the vectors are independent. Indeed, if we had  $\mathbf{a}_i = \sum_{j \in J} \mathbf{a}_j$  for some set  $J$  not containing  $i$ , then, on taking its inner product with  $\mathbf{a}_i$  we arrive at the contradiction that

$$1 = \langle \mathbf{a}_i, \mathbf{a}_i \rangle = \left\langle \mathbf{a}_i, \sum_{j \in J} \mathbf{a}_j \right\rangle = \sum_{j \in J} \langle \mathbf{a}_j, \mathbf{a}_i \rangle = \sum_{j \in J} 0 = 0.$$

Finally, since  $\mathbf{a}_1, \dots, \mathbf{a}_m$  are independent vectors in an  $n$ -dimensional vector space,  $m \leq n$ , as claimed. □

**Notes.** This is a well-known result proved independently by E.R. Berlekamp (On subsets with intersections of even cardinality, *Canad. Math. Bull.* **12** (1969), 471–474) and J.E. Graver (Boolean designs and self-dual matroids, *Linear Algebra and Appl.* **10** (1975), 111–128).

## 60. A Politically Correct Town

*Suppose that in a town of  $n$  men and  $n$  women, every club has the same number of women as men. All clubs have different sets of members and, for any two clubs, exactly as many women as men belong to both clubs. Then the maximum of the number of clubs is  $2^n$ .*

*Proof.* Writing  $\{x_1, \dots, x_n\}$  and  $\{y_1, \dots, y_n\}$  for the sets of men and women, if for every subset  $I$  of the index set  $[n]$  we form a club  $C_I = M_I \cup W_I$ , where  $M_I = \{x_i : i \in I\}$  and  $W_I = \{y_i : i \in I\}$ , then the set of these  $2^n$  clubs has the required property. (There is also an ‘empty club’ with no members.)

Conversely, every club  $C = M \cup W$  is determined by its male membership, since if we had another club  $C' = M \cup W'$  with the same male membership then we would have  $|W \cap W'| = |M| = |W| = |W'|$ , i.e., we would have  $W = W'$ , implying that the clubs  $C$  and  $C'$  have precisely the same members.  $\square$

## 61. Lattice Paths

The number of increasing lattice paths on  $\mathbb{Z}^2$  from  $(r, s)$  to  $(m, n)$  that never rise above the line  $y = x$  is

$$\binom{n+m-r-s}{m-r} - \binom{n+m-r-s}{m-s+1}.$$

*Proof.* Let  $\mathcal{L}$  be the set of increasing paths from  $(r, s)$  to  $(m, n)$ , and let  $\mathcal{L}'$  be the subset of those paths that do not go above the line  $y = x$ . Also, set  $\mathcal{L}'' = \mathcal{L} \setminus \mathcal{L}'$ , and write  $\mathcal{L}^*$  for the set of increasing paths from  $(s-1, r+1)$  to  $(m, n)$ . We claim that there is a simple one-to-one correspondence between  $\mathcal{L}''$  and  $\mathcal{L}^*$ . Indeed, given  $L \in \mathcal{L}''$ , let  $L_1$  be the initial segment of  $L$  ending in the first point of  $L$  above the line  $y = x$ , i.e., in a point of the form  $(x, x+1)$ . Let  $L^* \in \mathcal{L}^*$  be the path obtained from  $L$  by reflecting its initial segment in the line  $y = x+1$ , as shown in Figure 63. The map  $\mathcal{L}'' \rightarrow \mathcal{L}^*$ ,  $L \mapsto L^*$ , gives a one-to-one correspondence between the sets  $\mathcal{L}''$  and  $\mathcal{L}^*$ : the inverse is given by reflecting the initial segment of a path in  $\mathcal{L}^*$  ending in the first point on the line  $y = x+1$  in this line.

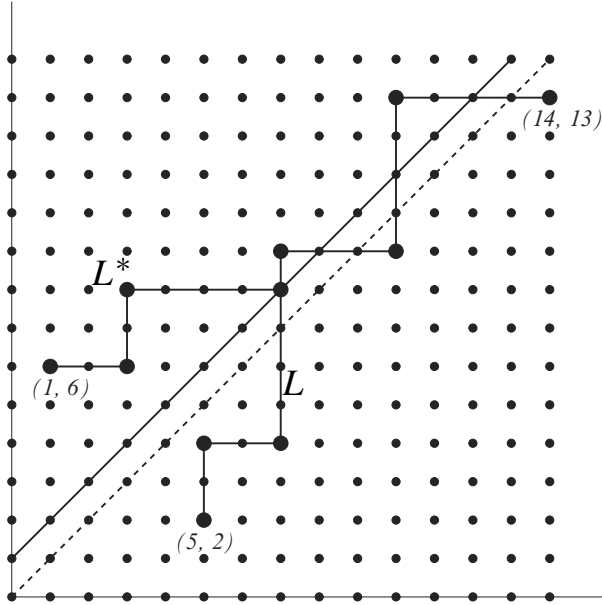


Fig. 63. A path  $L \in \mathcal{L}''$ , and the path  $L^* \in \mathcal{L}^*$  it defines.

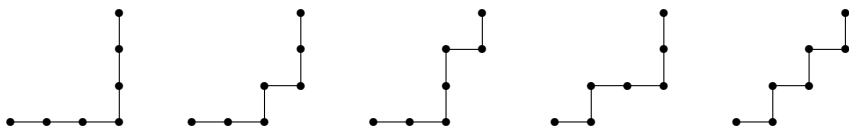


Fig. 64. The five lattice paths that increase monotonically from  $(0, 0)$  to  $(3, 3)$  and do not rise above the line  $y = x$ .

Consequently,  $|\mathcal{L}''| = |\mathcal{L}^*| = \binom{n+m-r-s}{m-s+1}$ , and so the answer to the question is

$$|\mathcal{L}'| = |\mathcal{L}| - |\mathcal{L}''| = |\mathcal{L}| - |\mathcal{L}^*| = \binom{n+m-r-s}{m-r} - \binom{n+m-r-s}{m-s+1}.$$

□

**Notes.** Although it seems that in this question we have four parameters,  $r$ ,  $s$ ,  $m$  and  $n$ , the answer does not change if each of the four quantities is increased by the same amount, so we may always assume that  $s = 0$ , say.

The answer is especially pleasing in the case  $r = s = 0$  and  $m = n$ : the number of increasing paths from  $(0, 0)$  to  $(n, n)$  that do not go above the line  $y = x$  is

$$\binom{2n}{n} - \binom{2n}{n+1} = \binom{2n}{n} \left(1 - \frac{n}{n+1}\right) = \frac{1}{n+1} \binom{2n}{n},$$

which is the  $n$ th Catalan number  $C_n$ . See Figure 64 for the case  $n = 3$ .

Putting it slightly differently, this result says that in the space of random increasing paths from  $(0, 0)$  to  $(n, n)$  the probability of never going above the line  $y = x$  is  $1/(n+1)$ .

Similarly, the number of increasing paths from  $(r, 0)$  to  $(n, n)$  that do not go above the line  $y = x$  is

$$\binom{2n-r}{n} - \binom{2n-r}{n+1} = \binom{2n-r}{n} \left(1 - \frac{n-r}{n+1}\right) = \frac{r+1}{n+1} \binom{2n-r}{n},$$

i.e., the probability that an increasing walk from  $(r, 0)$  to  $(n, n)$  does not go above the line  $y = x$  is  $(r+1)/(n+1)$ .

Monotone increasing walks on  $\mathbb{Z}^2$  correspond to standard walks on  $\mathbb{Z}$ : such an identification can be obtained, for example, by making a step  $(u, v) \rightarrow (u+1, v)$  correspond to a step  $x \rightarrow x+1$ , and a step  $(u, v) \rightarrow (u, v+1)$  to  $x \rightarrow x-1$ . Under this correspondence, the result proved in this question is the following.

Let  $s$ ,  $t$  and  $n$  be non-negative integers, with  $t \leq s+n$  and  $s+t+n$  even, say,  $s+t+n = 2m$ . Then the number of  $n$ -step walks on  $\mathbb{Z}$  from  $s$  to  $t$  which never go to zero is

$$\binom{n}{m-s} - \binom{n}{m+1}.$$

For  $s = t = 0$  this is the  $m$ th Catalan number:  $\frac{1}{m+1} \binom{2m}{m}$ .

Also, condition a random walk on  $\mathbb{Z}$  to start at  $s \geq 0$  and ending at 0 after  $n$  steps, where  $s + n$  is even, say,  $s + n = 2m$ . Then the probability that the walk never takes negative values is  $(s + 1)/(m + 1)$ .

Let us say a few more words about the case  $s = 0$ , leading to Catalan numbers. A *positive  $n$ -walk* is a walk on  $\mathbb{Z}$  that starts at 0 and, after  $2n$  steps, ends at 0, without ever taking negative values. We know that the number  $W_n$  of positive  $n$ -walks is precisely the  $n$ th Catalan number  $C_n = \frac{1}{n+1} \binom{2n}{n}$ , so that  $W_0 = 1$ ,  $W_1 = 1$ ,  $W_2 = 2$ ,  $W_3 = 5$ , and so on. The definition of a positive  $n$ -walk easily gives a recursion for  $W_n$  (and so for  $C_n$ ); this recursion is frequently useful in showing that a combinatorially defined number is a Catalan number.

Let  $n \geq 1$ . Clearly, every positive  $n$ -walk starts with a step  $0 \rightarrow 1$  and ends with a step  $1 \rightarrow 0$ . For a positive  $n$ -walk  $P_n$ , let  $2k$  be the first time that it returns to 0. Then  $P_n$  can be decomposed into a step  $0 \rightarrow 1$ , followed by a positive  $(k - 1)$ -walk shifted by one to the right, followed by a step  $1 \rightarrow 0$ , and ending with a positive  $(n - k)$ -walk. Conversely, every walk obtained in this way is a positive  $n$ -walk. Hence,

$$W_n = W_{n-1} + W_{n-2}W_1 + \cdots + W_1W_{n-2} + W_{n-1} = \sum_{i=0}^{n-1} W_i W_{n-1-i}.$$

This shows that the Catalan numbers  $C_n = \frac{1}{n+1} \binom{2n}{n}$  are determined by the condition  $C_0 = 1$  and the recursion

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} \quad (24)$$

valid for  $n \geq 1$ .

In fact, (24) and the boundary condition  $C_0 = 1$  easily imply the standard formula for  $C_n$ : writing  $f(X) = \sum_{n=0}^{\infty} C_n X^n$  for the generating function of the sequence  $(C_n)$ , we see from (24) and  $C_0 = 1$  that

$$f(X) = 1 + Xf(X)^2,$$

and so

$$f(X) = \frac{1 - \sqrt{1 - 4X}}{2X}.$$

Consequently,

$$C_n = (-1)^n \binom{1/2}{n+1} 4^{n+1}/2 = \frac{1}{n+1} \binom{2n}{n},$$

as claimed.

## 62. Triangulations of Polygons

There are  $C_n$  ways of triangulating a convex  $(n + 2)$ -gon by non-intersecting diagonals, where  $C_n = \frac{1}{n+1} \binom{2n}{n}$  is the  $n$ th Catalan number.

*Proof.* We shall prove the result by showing that the numbers satisfy the recurrence relation (24).

For simplicity, by a *triangulation* of a convex  $(n + 2)$ -gon we mean a triangulation into  $n$  triangles, i.e., using  $n - 1$  full diagonals. We shall write  $A_1 A_2 \dots A_{n+2}$  for our convex  $(n + 2)$ -gon  $P_{n+2}$ , and will consider the indices modulo  $n + 2$ , so that  $n + 3$  counts as 1,  $n + 4$  as 2, and so on.

Write  $T_n$  for the number of triangulations of  $P_{n+2} = A_1 A_2 \dots A_{n+2}$  into  $n$  triangles, so that  $T_1 = 1$ ,  $T_2 = 2$  and  $T_3 = 5$ , and so on. Also, define  $T_0$  to be 1.

Let  $n \geq 2$ . Note that for every triangulation of  $P_{n+2} = A_1 A_2 \dots A_{n+2}$  the side  $A_{n+2} A_1$  is in a unique triangle  $A_{n+2} A_1 A_k$ , where  $2 \leq k \leq n + 1$ . Since the polygon  $A_1 A_2 \dots A_k$  has  $T_{k-2}$  triangulations, and  $A_{n+2} A_k A_{k+1} \dots A_{n+1}$  has  $T_{n+1-k}$ , the triangle  $A_{n+1} A_1 A_k$  appears in  $T_{k-2} T_{n+1-k}$  triangulations. Hence,

$$T_n = \sum_{k=2}^{n+1} T_{k-2} T_{n+1-k} = \sum_{k=0}^{n-1} T_k T_{n-1-k}.$$

Since  $T_2 = 1$ , (24) of Problem 61 tells us that  $T_n = C_n$ , as claimed. □

**Notes.** This is a classical question, usually attributed to Leonhard Euler. For a number of related questions, see Problem 100.



### 63. A Converse of Cauchy's Inequality: Zagier's Inequality

Let  $\mathcal{D}$  be the set of decreasing functions  $f : [0, \infty) \rightarrow [0, \infty)$ , and  $\mathcal{U}$  the set of integrable functions  $F : [0, \infty) \rightarrow [0, 1]$ . Then, for  $f, g \in \mathcal{D}$  and  $F, G \in \mathcal{U}$ , we have

$$\langle f, g \rangle \max\{\|F\|_1, \|G\|_1\} \geq \langle f, F \rangle \langle g, G \rangle. \quad (25)$$

*Proof.* As we shall see, (25) is an easy consequence of the well known (and rather trivial) fact that for  $S > 0$  we have

$$\frac{1}{S} \int_0^S f(x)g(x)dx \geq \left( \frac{1}{S} \int_0^S f(x)dx \right) \left( \frac{1}{S} \int_0^S g(x)dx \right). \quad (26)$$

This inequality, which is frequently referred to as *Chebyshev's inequality*, is clearly the special special case of (25) with  $F(x) = G(x) = 1$  if  $0 \leq x \leq S$  and  $F(x) = G(x) = 0$  if  $x > S$ .

To start our proof, we may clearly assume that  $\|F\|_1 = \|G\|_1 = S$  for some  $S > 0$ . Note that, by (26),

$$\begin{aligned} \frac{1}{S} \int_0^\infty f(x)g(x)dx &\geq \frac{1}{S} \int_0^S f(x)g(x)dx \\ &\geq \left( \frac{1}{S} \int_0^S f(x)dx \right) \left( \frac{1}{S} \int_0^S g(x)dx \right). \end{aligned}$$

Also, since  $f$  is decreasing,

$$\max\{\langle f, F \rangle : F \in \mathcal{U}, \|F\|_1 = S\} = \int_0^S f(x)dx,$$

and an analogous relation holds for  $g$  and  $G$ . Therefore (25) holds.  $\square$

**Notes.** This simple inequality is due to Zagier. It is interesting to note that Zagier's first proof of this result (see J. Math. Econom. 12 (1983), 103–118) was far from trivial.

D. Zagier, A converse to Cauchy's inequality, *Amer. Math. Monthly* **102** (1995), 919–920.

## 64. Squares Touching a Square

*At most eight unit squares can touch another.*

*Proof.* We shall put a ‘rope’  $R$  around the unit square  $S$  to be touched by the others, and we shall show that each square touching  $S$  contains a substantial portion of  $R$ .

To this end, let  $R$  be the perimeter of the square of side-length 2 obtained by blowing up  $S$  from its centre, as in Figure 65. As the rope  $R$  has perimeter length 8, it suffices to show that every solid unit square  $T$  touching  $S$  (and disjoint from the interior of  $S$ ) contains at least length 1 of the rope. If  $R$  cuts across  $T$ , i.e., goes from a side of  $T$  to the opposite side, then there is nothing to prove. Otherwise, by sliding  $T$  along  $S$ , we may assume that  $S$  and  $T$  share a vertex and then (assuming, as before, that  $R$  does not cut across  $T$ ) the square contains exactly a unit length of the rope  $R$ , and we are done. See the squares in Figure 65 for an illustration of various possibilities.  $\square$

**Notes.** This problem was posed as a coffee time problem by Oleg Pikhurko.

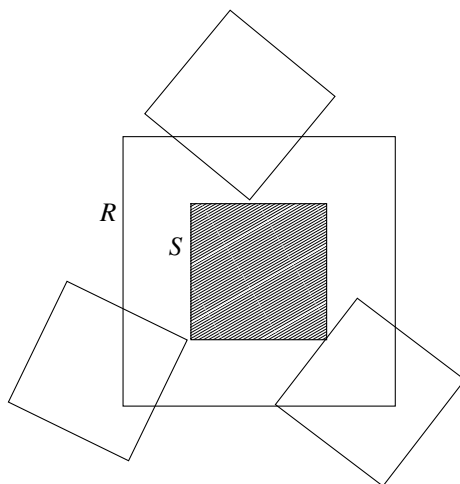


Fig. 65. A central square  $S$  with a ‘rope’  $R$  around it, together with three squares touching  $S$ .

## 65. Infection with Three Neighbours

*For  $n = 6k + 2$ , the minimal number of infected sites needed to spread the infection to all  $n^2$  sites of an  $n$  by  $n$  grid, provided a site becomes infected if at least three of its neighbours are infected, is  $(n + 1)^2/3 + 1$ .*

*Proof.* In view of the solution to Problem 34 we presented, the lower bound is very easily proved. Indeed, as in that solution, let us replace the grid by a chess board, so that the sites become the (unit) squares of this chess board: we call these unit squares *cells*. As earlier, colour an infected cell black and a healthy one white. Suppose that the infection process starts with  $s$  infected cells and, one by one, all  $t = n^2 - s$  other cells become infected. At the start of the process, the perimeter length of the infected region (the union of the infected squares) is at most  $4s$ , and at the end it is  $4n^2$ . This time, when a healthy cell gets infected, the perimeter length of the infected region *decreases* by at least 2, since at least three sides of the cell to be infected are adjacent to the infected region. Consequently,

$$4s - 2t = 4s - 2(n^2 - s) \geq 4n,$$

and so

$$s \geq \lceil n(n + 2)/3 \rceil.$$

For  $n$  even we can do a little better. To achieve full infection, the four cells in the corners of the grid must be infected, as they have only two neighbours, so they cannot get infected later. For a similar reason, we cannot have two *neighbouring* healthy cells on the boundary. Consequently, at the beginning of the process, each side contains two neighbouring infected cells. This implies that at the start of the process the perimeter length of the infected region is at most  $4s - 8$ , rather than the absolute upper bound  $4s$  we worked with above. Consequently, the inequality above is improved to

$$s \geq \lceil (n + 1)^2/3 \rceil + 1.$$

Turning to the upper bound, let us show that for  $n = 6k + 2$  this simple lower bound is attained, i.e., there is an arrangement of  $(n + 1)^2/3 + 1$  infected sites that eventually infects all the sites. This is perhaps most easily seen from Figure 66, where a suitable arrangement is given for  $k = 1$ .

With the aid of Figure 66, it is easy to describe the construction for  $n = 6k + 2$ , and any  $k$ . Pick every second site in the first row, starting with the first site, and every second site of the second, but shifted by one (and so ending with the last site of the row), and pick the first site of the third row. The arrangement in the next three rows is similar, but working backwards (i.e., interchanging right and left): in the fourth row we pick every second site, starting with the second (and so ending with

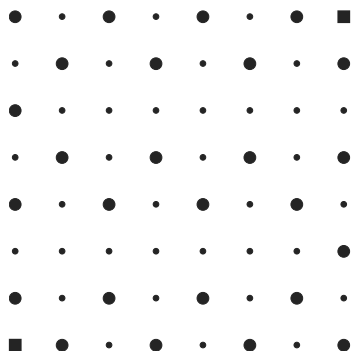


Fig. 66. The case  $k = 1$ ,  $n = 8$ . Starting with the above 28 infected sites of the  $8 \times 8$  board, eventually every site will be infected. The initially infected sites are the solid black dots, and the two corner sites added at the end are the black squares.

the last), in the fifth, we also pick every second site, but starting with the first, and we pick the last site of the sixth row. This pattern is repeated in every six rows, but the last two rows are like the first two. What we have so far *almost* works: every site gets infected but the top right corner and the bottom left corner. But we know that those must be infected at the very beginning, so we add them to the set of infected sites. In this way, we start with  $2k(3k + 1 + 3k + 1 + 1) + (3k + 1 + 3k + 1) + 2 = 12k^2 + 12k + 4 = (n + 1)^2/3 + 1$  infected sites, as required; it is trivial that eventually every site will be infected.  $\square$

**Notes.** Clearly, the lower bound  $n(n + 2)/3$  is very close to the actual minimum for every value of  $n$ , as variants of the construction above show.

## 66. The Spread of Infection on a Torus

*The minimal number of sites needed to infect eventually all  $n^2$  sites of an  $n$  by  $n$  torus is  $n - 1$ .*

*Proof.* We shall make use of what we have learned in the solutions of Problems 34 and 35 to show that the minimum is at least  $n - 1$ . In particular, by considering the perimeter length of the union of the infected cells in the square lattice formulation of the problem, as in Problem 34, we see that in an  $r \times s$  rectangle one needs at least  $(r + s)/2$  infected sites to infect the entire rectangle.

Let  $T_n$  be the  $n \times n$  torus formed by the  $n^2$  sites  $\{(i, j) : 0 \leq i, j \leq n - 1\}$ . That the minimum is at most  $n - 1$  is shown by the set  $\{(i, i) : 0 \leq i \leq n - 1\}$ , say. Our aim then is to prove the real assertion that if  $S \subset T_n$  infects  $T_n$  then  $|S| \geq n - 1$ .

The infection process can be described as follows. At the very beginning,  $S$  is the disjoint union of some rectangles of the form  $[a_1, b_1) \times [a_2, b_2)$ , with a half-open interval  $[a, b) = \{a, a + 1, \dots, b - 1\}$  taken modulo  $n$ . (In the solution of Problem 35 the high-dimensional rectangles were our *boxes*.) If no two boxes are within distance 2 (more precisely, within graph distance 2 in the torus graph), no new infection can occur. Otherwise, some two rectangles,  $R_s$  and  $R_t$ , say, are within distance 2, and so  $R_s \cup R_t$  infects the minimal rectangle  $R_{st} = R_s \vee R_t$  that contains both, and we continue the process with one fewer rectangles:  $R_{st}$  in place of  $R_s$  and  $R_t$ . Furthermore, if this rectangle  $R_{st}$  has a side of length  $n - 1$  then it infects a whole ‘band’ of the form  $\{(i, j) : a_1 \leq i < b_1, 0 \leq j \leq n - 1\}$  or  $\{(i, j) : 0 \leq i \leq n - 1, a_2 \leq j < b_2\}$ .

During the process, eventually we must have such a band: let us stop the process the moment a band appears. Suppose this is a  $k \times n$  band  $B$  of  $k$  full columns. Then  $B$  was created from a  $k \times (n - 1)$  rectangle  $R$ , so  $S$  has at least  $(k + n - 1)/2$  elements in  $R$ . Furthermore,  $S$  has at least one element in any two neighbouring columns, so it has at least  $(n - k - 1)/2$  elements outside  $R$ . Consequently,

$$|S| \geq \frac{k + n - 1}{2} + \frac{n - k - 1}{2},$$

as required. □

## 67. Dominating Sequences

Let  $\varphi$  be a strictly convex decreasing function on  $\mathbb{R}^+$ , and let  $\mathbf{x} = (x_i)_1^n$  and  $\mathbf{y} = (y_i)_1^n$  be strictly positive sequences such that  $\mathbf{x}$  is increasing and dominates  $\mathbf{y}$ . Then  $S_\varphi(\mathbf{x}) \leq S_\varphi(\mathbf{y})$ , with equality if and only if  $\mathbf{x} = \mathbf{y}$ .

*Proof.* Let us apply induction on  $n$ . For  $n = 1$  there is nothing to prove so we turn to the induction step. Let  $n > 1$  and suppose that the assertion holds for smaller values of  $n$ . Let  $\varphi$  be a function on  $(0, \infty)$  that is strictly convex and decreasing, and let  $\mathbf{x} \in M_n^*$  dominate  $\mathbf{y} \in M_n$ , where  $M_n$  is the set of strictly positive sequences of length  $n$ , and  $M_n^*$  is the subset of monotone increasing sequences.

If  $\sum_{i=1}^k x_i = \sum_{i=1}^k y_i$  for some  $k$ ,  $1 \leq k \leq n-1$ , then we are done by the induction hypothesis, since if we break  $\mathbf{x}$  into  $\mathbf{x}' = (x_i)_1^k$  and  $\mathbf{x}'' = (x_i)_{k+1}^n$ , and  $\mathbf{y}$  into  $\mathbf{y}' = (y_i)_1^k$  and  $\mathbf{y}'' = (y_i)_{k+1}^n$ , then we find that  $\mathbf{x}' \geq \mathbf{y}'$  and  $\mathbf{x}'' \geq \mathbf{y}''$ . Hence,  $S_\varphi(\mathbf{x}) = S_\varphi(\mathbf{x}') + S_\varphi(\mathbf{x}'') \leq S_\varphi(\mathbf{y}') + S_\varphi(\mathbf{y}'') = S_\varphi(\mathbf{y})$ , with equality if and only if  $\mathbf{x} = \mathbf{y}$ .

Otherwise,  $\sum_{i=1}^k x_i > \sum_{i=1}^k y_i$  for  $k = 1, \dots, n-1$ . Set

$$\delta = \min \left\{ \sum_{i=1}^k x_i - \sum_{i=1}^k y_i : k = 1, \dots, n-1 \right\},$$

and

$$\mathbf{x}' = (x_1 - \delta, x_2, \dots, x_{n-1}, x_n + \delta).$$

Then  $\mathbf{x}' \geq \mathbf{y}$  and  $\sum_{i=1}^k x_i = \sum_{i=1}^k y_i$  for some  $k$ ,  $1 \leq k \leq n-1$  so, as we have seen,  $S_\varphi(\mathbf{x}') \leq S_\varphi(\mathbf{y})$ . Furthermore, as  $\varphi$  is strictly convex,

$$S_\varphi(\mathbf{x}') - S_\varphi(\mathbf{x}) = \varphi(x_1 - \delta) - \varphi(x_1) + \varphi(x_n + \delta) - \varphi(x_n) > 0,$$

so  $S_\varphi(\mathbf{x}) < S_\varphi(\mathbf{x}') \leq S_\varphi(\mathbf{y})$ , as required. □

## 68. Sums of Reciprocals

Let  $(a_i)_1^n$  be a sequence of natural numbers such that all  $2^n - 1$  sums formed from the  $a_i$  are distinct. Then  $\sum_{i=1}^n 1/a_i < 2$ .

*Proof.* Let  $1 \leq a_1 < a_2 < \cdots < a_n$  be such that all  $2^n - 1$  sums  $\sum_{i \in I} a_i$  are distinct. In particular, there are  $2^k - 1$  distinct sums with summands  $a_1, \dots, a_k$ , and so the largest of these sums,  $a_1 + \cdots + a_k$ , is at least  $2^k - 1$ . Hence, setting  $b_i = 2^{i-1}$ ,  $i = 1, \dots, n$ , we find that

$$\sum_{i=1}^k a_i \geq 2^k - 1 = \sum_{i=0}^{k-1} 2^i = \sum_{i=1}^k b_i,$$

i.e., in the terminology of Problem 67, the increasing sequence  $(a_i)_1^n$  dominates the increasing sequence  $(b_i)_1^n$ . Since  $1/x$  is a convex function of  $x > 0$ , by Problem 67 we have

$$\sum_{i=1}^n 1/a_i \leq \sum_{i=1}^n 1/b_i = \sum_{i=1}^n 2^{1-i} = 2 - 2^{-n+1} < 2.$$

□

**Notes.** This question was set by Paul Erdős and Stan Benkoski in the *Canadian Mathematical Bulletin* **17** (1975), Problem 220, p. 768.

## 69. Absent-minded Passengers

*A fully booked plane is about to be boarded by its  $n \geq 2$  passengers. The first passenger loses his boarding card and sits down in a random place. From then on, whenever finding his seat taken, a passenger very politely takes a seat at random. Then the probability that the last passenger sits in his own seat is exactly  $1/2$ .*

*Proof.* Suppose there are  $n \geq 2$  passengers and  $n$  seats. If, for any  $k < n$ , the first  $k$  passengers occupy the seats assigned to them (as a group, not individually) then from then on everybody, including the last passenger, sits in his own seat. Also, if one of the first  $k < n$  passengers occupies the seat of the last passenger then the last passenger will not sit in his own seat. Now, if, for  $k - 1$ , neither of these events holds then the  $k$ th passenger's choice has exactly as much chance of leading to the first event as to the second: if his own seat is unoccupied, neither of the events will happen, and if his own seat is occupied then there is exactly one (unoccupied) seat, the first, that leads to the first event and exactly one (unoccupied) seat, the last, that leads to the second.  $\square$

**Notes.** This question was on *Car Talk*, the popular NPR program airing on Saturdays, and was brought to my attention by Oliver Riordan, who also gave the lovely proof above.

Note that a full plane is important. If there are  $n \geq 2$  passengers and  $N > n$  seats (a much more realistic assumption!) then it is more likely that the last passenger sits in his own seat.



## 70. Airline Luggage

Let  $\ell(B) = a + b + c$  be the linear measure of an  $a \times b \times c$  box (rectangular parallelepiped). This linear measure is monotone in the sense that if a box  $B$  is contained in a box  $C$  then  $\ell(B) \leq \ell(C)$ .

*First Proof.* The *Hint* is trying to guide us towards the following extremely elegant and unexpected solution.

For a set  $X$  in  $\mathbb{R}^3$  and a positive real number  $r$ , let  $X_{(r)}$  be the  $r$ -neighbourhood of  $X$ , the set of all points in  $\mathbb{R}^3$  within distance  $r$  of  $X$ . Also, let us write  $\text{vol}(X)$  for the volume of  $X$ . (To be precise, we use  $\text{vol}$  for the Lebesgue measure of a measurable set; as we shall see in a moment, our sets will be trivially measurable.)

Let  $B \subset C$  be boxes, say,  $B$  a  $b_1 \times b_2 \times b_3$  box and  $C$  a  $c_1 \times c_2 \times c_3$  box. Since  $B_{(r)}$  is contained in  $C_{(r)}$ ,

$$\text{vol}(B_{(r)}) \leq \text{vol}(C_{(r)}). \quad (27)$$

The volumes of  $B_{(r)}$  and  $C_{(r)}$  are easily computed. Indeed, the set  $B_{(r)}$  is the union of  $B$ , the set  $B_1$  formed by the six ‘slabs’ of height  $r$  sitting on the faces, the set  $B_2$  formed by quarter cylinders of radius  $r$  around the twelve edges, and the set  $B_3$  formed by eight octants of a ball of radius  $r$  about each vertex. It is difficult not to notice that the eight octants can be put together to form a ball of radius  $r$  and the volume of this ball is  $\frac{4\pi}{3}r^3$ ; nevertheless, this is irrelevant: all that matters is that the total volume of these octants is  $c_3r^3$  for some positive constant  $c_3$ . Similarly, the twelve quarter cylinders can be put together to form a cylinder of radius  $r$  and height  $b_1 + b_2 + b_3 = \ell(B)$  of volume  $\pi r^2 \ell(B)$ : again all that matters is that this volume is  $c_2 r^2 \ell(B)$  for some positive constant  $c_2$ . Finally, the volume of the slab over the face of side-lengths  $b_i$  and  $b_j$  is  $b_i b_j r$ . Consequently,  $\text{vol}(B_{(r)})$  is a simple cubic polynomial of  $r$ :

$$\text{vol}(B_{(r)}) = b_1 b_2 b_3 + 2r \sum_{1 \leq i < j \leq 3} b_i b_j + \pi r^2 \ell(B) + \frac{4\pi}{3} r^3, \quad (28)$$

and a similar expression holds for  $\text{vol}(C)$ . Hence, by (27), the coefficient of  $r^2$  in  $\text{vol}(B_{(r)})$ , which is precisely  $\pi \ell(B)$ , is at most the corresponding coefficient for  $\text{vol}(C_{(r)})$ , namely  $\pi \ell(C)$ , completing the solution.

It is easy to prove a little more: if a box  $B$  is contained in a box  $C$  and  $\ell(B) \geq \ell(C)$  then  $B = C$ . Indeed, letting  $r \rightarrow \infty$ , we see from (27) and (28) that  $\ell(B) = \ell(C)$ , and then that

$$b_1 b_2 + b_2 b_3 + b_3 b_1 \leq c_1 c_2 + c_2 c_3 + c_3 c_1.$$

These relations imply that

$$(b_1 + b_2 + b_3)^2 - (b_1^2 + b_2^2 + b_3^2) \leq (c_1 + c_2 + c_3)^2 - (c_1^2 + c_2^2 + c_3^2),$$

which is equivalent to

$$(b_1^2 + b_2^2 + b_3^2) \geq (c_1^2 + c_2^2 + c_3^2).$$

The last inequality says that the distance between two diametrically opposite points of the box  $B$  is at least as large as the distance between two diametrically opposite points of  $C$ . Hence, every vertex of  $B$  is also a vertex of  $C$ , i.e.,  $B = C$ , as claimed.  $\square$

*Second Proof.* Let the boxes  $B \subset C$  be as in the first proof, with  $B = \{\mathbf{x} = (x_1, x_2, x_3) : 0 \leq x_i \leq b_i, i = 1, 2, 3\}$ , say. Then the area of the surface of  $C$  above the top face of  $B$  is at least  $b_1 b_2$ , since every surface  $F$  of the form  $F = \{(x_1, x_2, f(x_1, x_2)) : 0 \leq x_i \leq b_i, i = 1, 2\}$  has area at least  $b_1 b_2$ . Similar assertions hold for other faces of the box  $B$ , so the surface area of  $C$  is at least as large as the surface area of  $B$ :

$$2b_1 b_2 + 2b_2 b_3 + 2b_3 b_1 \leq 2c_1 c_2 + 2c_2 c_3 + 2c_3 c_1,$$

with equality if and only if  $B = C$ . Since the diameter of  $C$  is at least as large as the diameter of  $B$ , we have

$$b_1^2 + b_2^2 + b_3^2 \leq c_1^2 + c_2^2 + c_3^2.$$

Adding these inequalities, we find that

$$(b_1 + b_2 + b_3)^2 \leq (c_1 + c_2 + c_3)^2,$$

with equality if and only if  $B = C$ , completing our solution.  $\square$

**Notes.** The problem seems to be folklore, as is its first, very witty solution. The second solution, which is even simpler and no less elegant, was found by Vladimir Nikiforov.

## 71. Intersecting Sets: the Erdős–Ko–Rado Theorem

Let  $\mathcal{A} \subset X^{(k)}$  be an intersecting family of  $k$ -subsets of an  $n$ -element set  $X$ , where  $2 \leq 2k \leq n$ . Then

$$|\mathcal{A}| \leq \binom{n-1}{k-1}.$$

*Proof.* We claim that if  $X = \mathbb{Z}_n$ , i.e., the elements of  $X$  are  $1, 2, \dots, n$  taken modulo  $n$ , then the number of  $k$ -intervals in  $\mathcal{A}$ , i.e., sets of the form  $\{a, a+1, \dots, a+k-1\}$ , is at most  $k$ . Although experience shows that people prefer to show this for themselves, we shall spell out an argument. Given a  $k$ -interval  $A \in \mathcal{A}$ , there are  $2k-2$  other  $k$ -intervals meeting  $A$ , namely  $B_1, \dots, B_{k-1}$ , with  $B_i \cap A = \{1, \dots, i\}$ , and  $C_1, \dots, C_{k-1}$ , with  $C_i \cap A = \{i+1, \dots, k\}$ . Since  $n \geq 2k$ , we have  $B_i \cap C_i = \emptyset$ , so  $\mathcal{A}$  contains at most one interval from each pair  $(B_i, C_i)$ . Hence  $\mathcal{A}$  contains at most  $k$  intervals, as claimed.

To complete the proof, we endow  $X$  with a random cyclic order, and compute the expected number of  $k$ -intervals in  $\mathcal{A}$ . Rather formally, let  $\pi : X \rightarrow \mathbb{Z}_n$  be a random one-to-one map. Write  $f_{\mathcal{A}}(\pi)$  for the number of sets in  $\mathcal{A}$  that are mapped into intervals. We have just shown that  $f_{\mathcal{A}}(\pi) \leq k$  for every map  $\pi$ . Also, the probability that a  $k$ -set  $A \in \mathcal{A}$  is mapped into an interval by a random map  $\pi$  is  $n/\binom{n}{k}$ , since of the  $\binom{n}{k}$  equally probable images of  $A$ , precisely  $n$  are intervals. Consequently,

$$\mathbb{E}(f_{\mathcal{A}}(\pi)) = \frac{n|\mathcal{A}|}{\binom{n}{k}} \leq k,$$

and so  $|\mathcal{A}| \leq \binom{n-1}{k-1}$ , completing the proof.  $\square$

**Notes.** This assertion is a fundamental and extremely influential result in extremal set theory, the *Erdős–Ko–Rado theorem*; it was proved by Paul Erdős, Chao Ko and Richard Rado when they were in Manchester in the 1930s, but was published only in 1961. The beautiful proof above, a real gem of a combinatorial argument, was given by Katona in 1972. Since then, *Katona’s circle method* has found many other applications.

P. Erdős, C. Ko, C., and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford Ser. (2)* **12** (1961), 313–320.

G.O.H. Katona, A simple proof of the Erdős–Chao Ko–Rado theorem, *J. Combin. Theory Ser. B* **13** (1972), 183–184.

## 72. Sperner Families: the MYBL Inequality

Let  $\mathcal{A}$  be a Sperner family of subsets of  $[n]$ . Then

$$\sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1} \leq 1,$$

with equality if and only if for some  $k$ ,  $0 \leq k \leq n$ ,  $\mathcal{A} = [n]^{(k)}$ , i.e.,  $\mathcal{A}$  consists of all  $k$ -subsets of  $[n]$ .

*First Proof.* For  $A \in \mathcal{A}$ , let  $S_A$  be the set of permutations  $p_1 p_2 \dots p_n$  of  $[n]$  of which  $A$  is an initial segment:  $A = \{p_1, p_2, \dots, p_{|A|}\}$ . Note that

$$|S_A| = |A|!(n - |A|)!,$$

and for  $A, B \in \mathcal{A}$  the sets  $S_A$  and  $S_B$  are disjoint. Since there are  $n!$  permutations, we find that

$$\sum_{A \in \mathcal{A}} |S_A| = \sum_{A \in \mathcal{A}} |A|!(n - |A|)! \leq n!,$$

that is

$$\sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1} \leq 1,$$

as claimed.

It is clear that we have equality if, for some  $k$ ,  $\mathcal{A}$  consists of all  $\binom{n}{k}$   $k$ -subsets of  $[n]$ . That in every other case we have strict inequality will be shown in the second proof.  $\square$

*Second Proof.* We shall prove the analogue of the MYBL inequality for ‘nice’ (finite, ranked, regular and connected) posets, partially ordered sets. In fact, that result is immediate from a simple fact concerning neighbourhoods of sets in bipartite graphs.

**Claim 1.** Let  $G$  be a connected  $(r, s)$ -regular bipartite graph with bipartition  $(U, W)$ . (Thus, every vertex in  $U$  has degree  $r$  and every vertex in  $W$  has degree  $s$ ; consequently,  $r|U| = s|W|$ .) Let  $A \subset U$ ,  $\emptyset \neq A \neq U$ . Then

$$|\Gamma(A)|/|W| > |A|/|U|, \quad (29)$$

where  $\Gamma(A)$  is the set of neighbours of  $A$ , i.e., the set of vertices in  $W$  adjacent to at least one vertex in  $A$ .

*Proof of Claim 1.* Let  $H$  be the subgraph of  $G$  spanned by  $A \cup \Gamma(A)$ . Then, as at least one vertex in  $\Gamma(A)$  has a neighbour not in  $A$ ,

$$r|A| = e(H) < s|\Gamma(A)|,$$

so

$$|A|/|U| = r|A|/s|W| < s|\Gamma(A)|/s|W| = |\Gamma(A)|/|W|.$$

□

From here, it is but a short step to the rather general (and almost trivial) inequality about posets: all we have to do is to set the scene. A set of incomparable elements in a poset is an *antichain*. Thus, in the power set  $\mathcal{P}(n)$  with the natural partial order ( $A < B$  if  $A \subset B$ ), an antichain is precisely a Sperner family of subsets.

A poset  $(P, <)$  is *ranked* if the vertex set is the union of *levels*  $L_0, L_1, \dots, L_n$ , say, and the order is induced by the relations  $x < y$  for  $x \in L_i$  and  $y \in L_{i+1}$ . More precisely:

(i) if  $x_i \in L_i$  then  $x_i < y_{i+1}$  for some  $y_{i+1} \in L_{i+1}$ , provided  $i < n$ , and  $x_i > z_{i-1}$  for some  $z_{i-1} \in L_{i-1}$ , provided  $i > 0$ ;

(ii) if  $x_i \in L_i$  and  $x_j \in L_j$  then  $x_i < x_j$  if and only if  $i < j$  and there are  $x_{i+1} \in L_{i+1}, \dots, x_{j-1} \in L_{j-1}$  such that  $x_i < x_{i+1} < \dots < x_{j-1} < x_j$ .

Let us note an immediate consequence of Claim 1 about shadows of sets in these posets. For  $K_i \subset L_i$  and  $j > i$ , the *shadow* of  $K_i$  at level  $j$  is

$$S_j(K_i) = \{y \in L_j : x < y \text{ for some } x \in K_i\}.$$

Note that for  $i < j < k$  and  $K_i \subset L_i$  we have  $S_k(S_j(K_i)) = S_k(K_i)$ . Claim 1 implies that if  $\emptyset \neq K_i \neq L_i$  then

$$|S_{i+1}(K_i)|/|L_{i+1}| > |K_i|/|L_i|. \quad (30)$$

At last, we can state the promised immediate consequence of Claim 1 extending the MYBL inequality.

**Claim 2.** *Let  $(P, <)$  be a finite, regular, locally connected ranked poset with level sets  $L_0, L_1, \dots, L_n$ . Let  $A$  be an antichain in  $(P, <)$ , and set  $A_i = A \cap L_i$ . Then*

$$\sum_{i=0}^n |A_i|/|L_i| \leq 1, \quad (31)$$

*with equality if and only if  $A = L_i$  for some  $i$ .*

*Proof of Claim 2.* If  $A_i = L_i$  for some  $i$  then  $A = L_i$ , and we have equality in (31). Otherwise, assuming that  $A \neq \emptyset$ , let  $m = \min\{i : A_i \neq \emptyset\}$ , so that  $\emptyset \neq A_m \neq L_m$ .

Set  $B_m = A_m$  and define  $B_{m+1}, B_{m+2}, \dots, B_n$  by setting  $B_{i+1} = S_{i+1}(B_i) \cup A_{i+1}$ . Since  $A$  is an antichain,  $S_j(A_i) \cap A_j = \emptyset$  for  $i < j$ , so the sets  $S_{i+1}(B_i)$  and  $A_{i+1}$  are disjoint. Consequently, by (30),

$$\begin{aligned} \sum_{i=0}^n |A_i|/|L_i| &= \sum_{i=m}^n |A_i|/|L_i| < |B_{m+1}|/|L_{m+1}| + \sum_{i=m+2}^n |A_i|/|L_i| \\ &\leq |B_{m+2}|/|L_{m+2}| + \sum_{i=m+3}^n |A_i|/|L_i| \leq \dots \leq |B_n|/|L_n| = 1. \end{aligned}$$

To complete our second proof of the original assertion (with equality), note that the power set  $\mathcal{P}(n)$  is a ranked regular poset, with  $[n]^{(i)}$ , the collection of all  $\binom{n}{i}$   $i$ -element sets forming level  $i$ ; indeed, every set (element) at level  $i$  contains  $i$  sets at level  $i-1$  and is contained in  $n-i$  sets at level  $i+1$ . All that remains to note is that every graph formed by two neighbouring levels is connected since, by induction, the bipartite graph formed by the sets  $[k]^{(i)}$  and  $[k]^{(i+1)}$  is connected.  $\square$

**Notes.** Since the maximum of  $\binom{n}{k}$  is attained for  $k = \lfloor n/2 \rfloor$  and  $\lceil n/2 \rceil$ , a Sperner family of subsets of an  $n$ -element set has at most  $\binom{n}{\lfloor n/2 \rfloor}$  elements. This is the content of *Sperner's Lemma*, proved in 1928. The inequality in this problem was discovered several times: Yamamoto proved it in 1954, Meshalkin in 1963, Bollobás (in a considerably stronger form, see Problem 104) in 1965, and Lubell in 1966. Strangely, it is often referred to as the LYM inequality. Eric Milner suggested calling it the LYMB inequality, and Gyula Katona recommended the acronym we have used, MYBL. There are several advantages of this usage: it accommodates all four authors, rather than only three; and it spells the name of one of the most famous architects in Hungary in the 19th century, *Miklós Ybl* (though not in the Hungarian order!), the architect of the magnificent Opera House and many beautiful palaces in Budapest. In the chronologically correct order the acronym would be YMBL, a culturally much less significant acronym!

The case of equality (again in a considerably more general form) was proved by Bollobás in 1965, although the proof above is from 1986.

B. Bollobás, On generalized graphs, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 447–452.

B. Bollobás, *Combinatorics – Set Systems, Hypergraphs, Families of Vectors, and Combinatorial Probability*, Cambridge University Press (1986), 190 pp.

D. Lubell, A short proof of Sperner's lemma, *J. Combin. Theory* **1** (1966), 299.

L.D. Meshalkin, A generalization of Sperner's theorem on the number of subsets of a finite set (in Russian, with summary in German), *Teor. Veroyatnost. i Primenen* **8** (1963), 219–220.

E. Sperner, Ein Satz Über Untermengen einer endlichen Menge, *Math. Z.* **27** (1928), 544–548.

K. Yamamoto, Logarithmic order of free distributive lattice, *J. Math. Soc. Japan* **6** (1954), 343–353.

### 73. Breaking a Stick

Divide the interval  $[0, 1]$  into  $n + 1$  segments by  $n \geq 2$  points chosen at random. The probability that these  $n + 1$  segments can be put together to form a convex  $(n + 1)$ -gon is  $1 - (n + 1)2^{-n}$ .

*Proof.* Let  $X_1, \dots, X_n$  be the points chosen, with  $0 < X_1 < X_2 < \dots < X_n < 1$ . (Here and in what follows we shall ignore events of probability 0.) Let  $E$  be the event that the  $n+1$  segments *do not* form a convex  $(n+1)$ -gon, and note that  $E$  holds if and only if  $[0, 1]$  has a subinterval of length  $1/2$  containing no point  $X_i$ . Set  $X_0 = 0$ , and for  $i = 0, 1, \dots, n$ , let  $E_i$  be the event that  $I_i = [X_i, X_i + 1/2] \subset [0, 1]$  and  $X_{i+1} \notin I_i$ . Then our event  $E$  is the disjoint union of the events  $E_0, E_1, \dots, E_n$ .

Clearly,  $E_i$  holds if and only if

$$X_0 < \dots < X_i < X_{i+1} - 1/2 < X_{i+2} - 1/2 < \dots < X_n - 1/2 \leq 1/2.$$

Equivalently, a sequence  $(X_j)_1^n$  belongs to  $E_i$  if it is obtained by choosing each of our  $n$  random points from  $[0, 1/2]$  rather than  $[0, 1]$ :  $Y_1 < Y_2 < \dots < Y_n$ , say, and setting

$$X_j = \begin{cases} Y_j & \text{if } j \leq i, \\ Y_j + 1/2 & \text{if } j > i. \end{cases}$$

This shows that  $\mathbb{P}(E_i) = 2^{-n}$  for every  $i$ ,  $i = 0, 1, \dots, n$ , and so  $\mathbb{P}(E) = (n + 1)2^{-n}$ . □

**Notes.** This is folklore: it must have been rediscovered many times.

## 74. Triads

*The plane cannot be partitioned into pairwise disjoint triads.*

*Proof.* Recall that a *triad* is a subset of the plane homeomorphic to the union of the three straight-line segments joining the origin  $(0, 0)$  to the points  $(1, 0)$ ,  $(0, 1)$  and  $(-1, 0)$ , say. Thus a triad has a *centre* and three internally disjoint *branches*, with each branch joining the centre to an *end-point*.

Having made sure that we know what a triad is, let us turn to the problem. We shall prove that the plane does not contain an uncountable family of pairwise disjoint triads. This is clearly more than required by the problem since each triad is nowhere dense in the plane so by the Baire Category Theorem the plane is not the union of countably many triads.

Suppose, for a contradiction, that  $\{T_\gamma : \gamma \in \Gamma\}$  is an uncountable family of pairwise disjoint triads. We shall show that this assumption contradicts the fact that all the triads are in the plane.

An *arc-triple* is a triple  $(I_1, I_2, I_3)$  in which  $I_1, I_2$  and  $I_3$  are disjoint rational arcs of a rational circle  $C$ . Thus, the radius of the circle  $C$  is rational, and so are the coordinates of its centre; in addition, the angles of the end-points of the arcs  $I_i$  are rational multiples of  $\pi$ . We say that a triad  $T$  *crosses* an arc-triple  $A = (I_1, I_2, I_3; C)$  if the centre of  $T$  is within  $C$ , and the first point of the  $j$ th branch of  $T$  (emanating from its centre) that is on  $C$  is on  $I_j$ ; see Figure 67. Trivially,

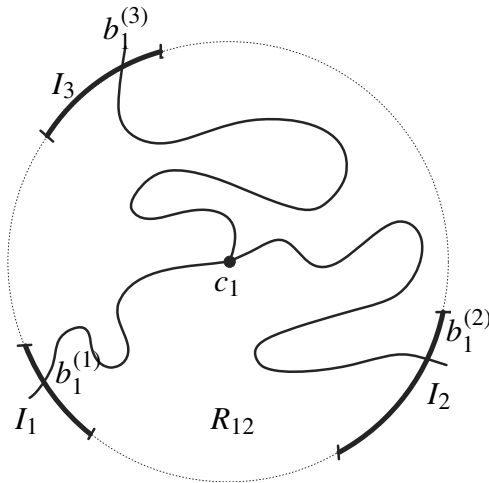


Fig. 67. A triad crossing an arc-triple.



each triad crosses at least one arc-triple (in fact, infinitely many arc-triples); since there are only countably many arc-triples, some arc-triple  $A = (I_1, I_2, I_3)$  crosses uncountably many triads. In fact, all we shall need is that  $A$  crosses two triads,  $T_1$  and  $T_2$ , say. Let  $C$  be the circle containing the arcs  $I_j$ ,  $D$  the disc bounded by  $C$ , and let the sub-branch of  $T_i$  from its centre  $c_i$  to the arc  $I_j$  meet the circle  $C$  (and so the arc  $I_j$ ) in the point  $b_i^{(j)}$ . The three sub-branches of  $T_1$  from  $c_1$  to the points  $b_1^{(j)}$  divide the disc  $D$  into three regions. We may assume that the centre  $c_2$  of  $T_2$  is in the region  $R_{12}$  bounded by the sub-branches  $c_1b_1^{(1)}$  and  $c_1b_1^{(2)}$ , and the arc  $b_1^{(1)}b_1^{(2)}$  of  $C$  not containing (and so not meeting)  $I_3$ . As  $b_2^{(3)}$  is on  $I_3$ , and so outside  $R_{12}$ , the sub-branch of  $T_2$  from  $c_2$  to  $b_2^{(3)}$  meets a branch of  $T_1$ . This contradiction completes our proof.  $\square$

**Notes.** This beautiful problem is certainly an old chestnut: I heard it in Budapest when I was an undergraduate.

Let us note the following slightly different way of completing the proof. Let  $J_j$  be the sub-arc of  $I_j$  from  $b_1^{(j)}$  to  $b_2^{(j)}$ . Let  $c_i$  be the centre of  $T_i$ , and let  $c_3$  be a point outside  $C$ . Join each  $J_i$  to  $c_3$  by a polygonal path  $P_i$  entirely outside  $C$ , with the exception of its end-point  $b_i$  on  $J_i$ , such that the paths are internally disjoint. Then the points  $c_1, c_2, c_3$  are joined by internally disjoint paths to the points  $b_1, b_2, b_3$ , giving us a planar drawing of the complete bipartite graph  $K_{3,3}$ . As  $K_{3,3}$  is non-planar (an immediate consequence of Euler's formula), this is a contradiction.

## 75. Colouring Complete Graphs

*Suppose that the complete graph  $K_n$ ,  $n \geq 4$ , has a proper edge-colouring with  $n - 1$  colours such that any six edges forming a complete graph on four vertices get either three colours or six. Then  $n = 2^m$  for some integer  $m \geq 2$ . Furthermore, for every such  $n$  there is such a colouring.*

*Proof.* Let  $a, b, \dots$  be the  $n - 1$  colours of the edges. Let us add a *fictitious* colour 0 to the set of colours, and let us define an *addition* on the set  $S = \{0, a, b, \dots\}$  of  $n$  colours as follows.

Note first that if there is a triangle whose edges get colours  $a, b$  and  $c$ , then every triangle with two edges coloured  $a$  and  $b$  is such that the third edge is coloured  $c$ . In this case, define  $a + b = c$ . Also, for every colour  $a$ , set  $a + a = 0$  and  $a + 0 = a$ . This addition is associative:  $(a + b) + c = a + (b + c)$ , since  $(a + b) + c$  and  $a + (b + c)$  are both the colour of the edge 14 when 12 has colour  $a$ , 23 has colour  $b$  and 34 has colour  $c$ . Hence, our addition turns  $S$  into an Abelian group of characteristic 2, so  $n$  is indeed a power of 2.

To see the converse, take the complete graph with vertex set  $\mathbb{Z}_2^m$  and colour an edge  $ab$ ,  $a, b \in \mathbb{Z}_2^m$  with  $a - b$ . □

**Notes.** This exceptionally beautiful proof (clearly from ‘The Book’) is from P. Keevash and B. Sudakov, On a hypergraph Turán problem of Frankl, *Combinatorica*, **25** (2005), 673–706.

## 76. Symmetric Convex Domains: a Theorem of Besicovitch

For a convex domain  $\Delta \subset \mathbb{R}^2$  of area  $a$ , let  $\alpha a$  be the maximal area of a centrally symmetric convex domain contained in  $\Delta$ . We call  $\alpha = \alpha(\Delta)$  the measure of symmetry of  $\Delta$ . Show that the infimum (really, minimum) of the measure of symmetry of a bounded convex planar domain is  $2/3$ .

*Proof.* (i) We show first that  $\alpha(\Delta) \geq 2/3$  for every convex planar domain  $\Delta$ . We may assume that  $\Delta$  is strictly convex, and  $\Gamma$ , its boundary, is differentiable. Indeed, the full assertion follows from this by a trivial compactness argument.

Given a direction  $\vartheta$ , take three chords of  $\Gamma$  with direction  $\vartheta$ , with the two outer ones,  $AB$  and  $DE$ , having the same length, and the middle one,  $CF$ , equidistant from them. With a slight abuse of notation, we shall write  $UV$  for the length of a segment  $UV$ . Note that the segments  $AD$ ,  $BE$  and  $CF$  meet in a point  $O$ , which is the midpoint of the segments  $AD$  and  $BE$ , but not necessarily of  $CF$ . Moving from the position in which  $A = B$  to the one in which all three chords coincide, we find a unique triple of chords with  $CF = 2AB$ . Clearly, the points  $A = A(\vartheta)$ ,  $B = B(\vartheta)$ ,  $\dots$ ,  $O = O(\vartheta)$  can be taken to be continuous functions of  $\vartheta$ . As  $\vartheta$  increases from 0, say, to  $\pi$ , opposite points get interchanged:  $A(0) = D(\pi)$ ,  $B(0) = E(\pi)$ ,  $\dots$ ,  $O(0) = O(\pi)$ . Therefore, there is a direction  $\vartheta_0$  such that  $O(\vartheta_0)$  is the midpoint of  $C(\vartheta_0)F(\vartheta_0)$ . These are the points we have been looking for: from now on, we write  $A$  for  $A(\vartheta_0)$ , and so on, as in Figure 68, where we have taken  $\vartheta_0 = 0$ .

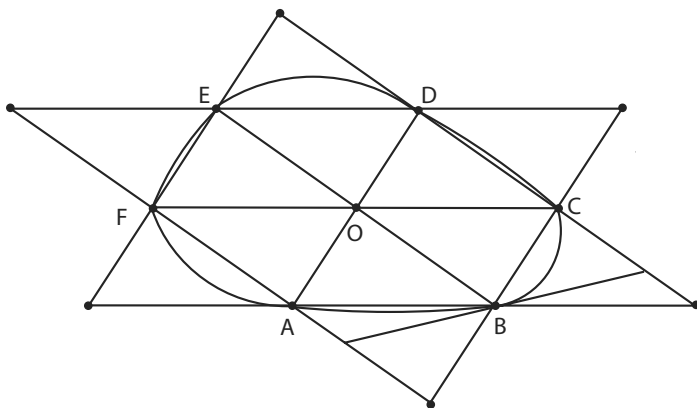


Fig. 68. The hexagon  $ABCDEF$  inscribed in the convex curve  $\Gamma$ .

We claim that the area of this hexagon  $ABCDEF$  is at least  $2a/3$ . To see this, note that the convex set  $\Delta$  is contained in the union of the twelve congruent triangles shown in Figure 68, and the hexagon is made up of six of these triangles. Furthermore,  $\Delta$  is the union of this hexagon, and six half-moon shaped regions  $H_{AB}, H_{BC}, \dots, H_{FA}$  on the chords  $AB, BC, \dots, FA$ . Taking a tangent of  $\Delta$  at  $B$ , say, we see that the sum of the areas of the half-moons  $H_{AB}$  and  $H_{BC}$  is at most the area of a triangle. The same applies to any two neighbouring half-moons. Consequently, the sum of the areas of the six half-moons is at most three times the area of a triangle. Since  $\Delta$  is contained in nine triangles, the area of a triangle is at least  $a/9$ , and so the area of the centrally symmetric convex hexagon  $ABCDEF$  is at least  $6a/9 = 2a/3$ . This proves that  $\alpha(\Delta) \geq 2/3$ , as required.

(ii) We claim that  $\alpha(\Delta) = 2/3$  if  $\Delta$  is a triangle. Indeed, let  $\Phi$  be a centrally symmetric convex set with centre  $O$  contained in the triangle. Then  $\Phi$  is contained in the intersection of the original triangle and its reflection in  $O$ . This intersection, which is either a parallelogram or a hexagon (see Figure 69), is easily seen to have area at most  $2a/3$ , where  $a$  is the area of  $\Delta$ . Indeed, the area of this intersection is maximal if  $O$  is the barycentre of the triangle.  $\square$

**Notes.** In Problem 6 we proved the rather trivial assertion that every convex polygon (or domain) of area 1 is contained in a rectangle of area 2; in particular, every convex domain of area 1 is contained in a centrally symmetric convex domain of area 2. The result in our problem above is the dual of this trivial assertion; this was proved independently by Besicovitch and Fáry. In a follow-up paper, Besicovitch showed that for convex domains  $\Delta$  of constant width the minimum of  $\alpha(\Delta)$

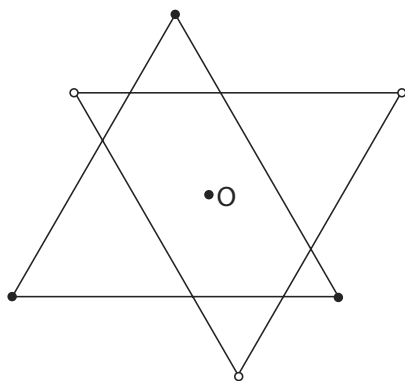


Fig. 69. A hexagonal intersection of a triangle  $\Delta$  with its reflection. Note that all eight triangles appearing are similar, and the sum of the perimeter lengths of the three 'small' triangles cut off a 'big' triangle by its reflection is precisely the perimeter length of a big triangle. As the measure of symmetry is invariant under affine transformations, we have taken  $\Delta$  to be regular.

is attained on a Rouleaux triangle; in particular, this minimum is approximately 0.840. As  $\alpha(\Delta) = 1$  when  $\Delta$  is symmetric and much smaller when  $\Delta$  is far from being symmetric, with apologies to Besicovitch, we called  $\alpha(\Delta)$  the measure of *symmetry* of  $\Delta$ , rather than the measure of its asymmetry.

A.S. Besicovitch, Measure of asymmetry of convex curves, *J. London Math. Soc.* **23** (1948), 238–240.

A.S. Besicovitch, Measure of asymmetry of convex curves (II): Curves of constant width, *J. London Math. Soc.* **26** (1951), 81–93.

I. Fáry, Sur la densité des réseaux de domaines convexes, *Bull. Société Mathématique de France* **78** (1949), 152–161.



## 77. Independent Random Variables

Let  $\mathcal{S} = \{S_1, S_2, \dots\}$  be a collection of finite sets of natural numbers. For  $i = 1, 2, \dots$ , let  $\mathbf{s}_i \in \mathbb{F}_2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$  be the characteristic vector of  $S_i$ :  $\mathbf{s}_i = (s_{i1}, s_{i2}, \dots)$ , where

$$s_{ij} = \begin{cases} 1 & \text{if } j \in S_i, \\ 0 & \text{if } j \notin S_i, \end{cases}$$

and  $\mathbb{F}_2$  is the field of two elements, 0 and 1. Note that the vectors  $\mathbf{s}_i$  are in the infinite-dimensional vector space  $V$  over  $\mathbb{F}_2$  consisting of all 0-1 sequences with finitely many 1s, with coordinate-wise operations.

Furthermore, let  $X_1, X_2, \dots$  be independent  $\pm 1$  random variables with  $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$  for every  $i$ , and set

$$Z_i = \prod_{j \in S_i} X_j.$$

Then the random variables  $Z_1, Z_2, \dots$  are independent if and only if the vectors  $\mathbf{s}_1, \mathbf{s}_2, \dots$  are too.

*Proof.* To establish the connection between the random variables  $Z_i$  and the vectors  $\mathbf{s}_j$ , for  $\mathbf{v} \in V$ , let  $\mathbf{X}^{\mathbf{v}}$  be the product of the random variables  $X_i$  with  $v_i = 1$ , where  $\mathbf{v} = (v_1, v_2, \dots)$ , so that each  $v_i$  is 0 or 1. Clearly, for any two vectors  $\mathbf{v}_1, \mathbf{v}_2 \in V$ , we have  $\mathbf{X}^{\mathbf{v}_1 + \mathbf{v}_2} = \mathbf{X}^{\mathbf{v}_1} \mathbf{X}^{\mathbf{v}_2}$ , and, by definition,  $Z_i = \mathbf{X}^{\mathbf{s}_i}$ .

It clearly suffices to show that every *finite* sequence of variables,  $Z_1, Z_2, \dots, Z_n$ , say, is independent if and only if the corresponding vectors  $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n$  are independent. Suppose first that  $\mathbf{s}_1, \dots, \mathbf{s}_n$  are dependent, i.e.,  $\sum_{i \in I} \mathbf{s}_i = \mathbf{0}$  for some non-empty index set  $I \subset [n]$ . But then

$$\prod_{i \in I} Z_i = \prod_{i \in I} \mathbf{X}^{\mathbf{s}_i} = \mathbf{X}^{\sum_{i \in I} \mathbf{s}_i} = \mathbf{X}^{\mathbf{0}} = 1.$$

In particular,  $\mathbb{P}(\prod_{i \in I} Z_i = 1) = 1$ , so the random variables  $Z_i$ ,  $i \in I$ , are not independent.

Conversely, suppose that the vectors  $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n$  are independent. We may and shall assume that the support of these vectors is exactly  $[n]$ . Indeed, if the support of these vectors is contained in  $[N]$ , say, we may extend the sequence  $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n$  to a basis  $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_N$  of  $\{0, 1\}^N$ ; writing  $n$  for  $N$ , our assumption is justified.

Let  $A$  be the  $n \times n$  matrix whose  $i$ th row is  $\mathbf{s}_i$ . Since the  $\mathbf{s}_i$  are independent, the linear map  $A : \{0, 1\}^n \rightarrow \{0, 1\}^n$  is one-to-one (mapping a finite set with  $2^n$  elements into itself).

This means exactly that the  $n$ -vector random variable  $\mathbf{Z} = (Z_1, Z_2, \dots, Z_n)$  takes every value in  $\{-1, +1\}^n$ , and every value comes from precisely one value of the vector  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ . In particular, the probability that  $\mathbf{Z}$  takes a particular value in  $\{-1, +1\}^n$  is exactly  $2^{-n}$ , i.e., the vectors  $Z_1, Z_2, \dots, Z_n$  are independent.  $\square$

**Notes.** The first part of the original question was brought to Memphis by Alexander Scott. To answer it, note that with  $S_1 = \{1, 2, 3\}$ ,  $S_2 = \{2, 3, 4\}$ ,  $\dots$ , the vectors  $\mathbf{s}_1, \mathbf{s}_2, \dots$  are clearly independent; therefore the random variables  $Y_1, Y_2, \dots$  are also independent, where  $Y_i = X_i X_{i+1} X_{i+2}$ .

## 78. Triangles Touching a Triangle

*The triangular lattice shows twelve congruent triangles touching a thirteenth (see Figure 70). There cannot be thirteen triangles touching a fourteenth.*

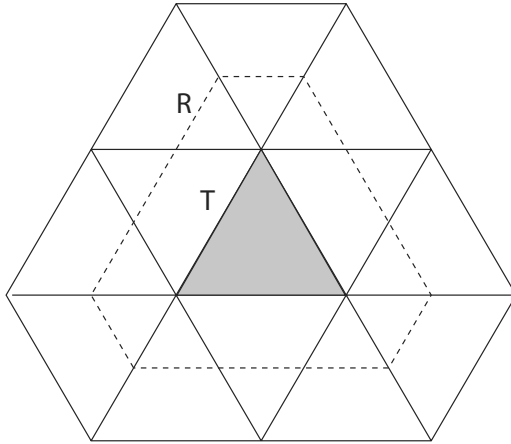


Fig. 70. A triangle  $T$  touched by twelve others, and the rope  $R$  surrounding it.

*Proof.* To show that the maximum is indeed twelve, apply the ‘rope trick’ of Problem 64. This time a suitable rope  $R$  is the hexagon connecting appropriate midpoints of the sides of the surrounding triangles, as in Figure 70. Then  $R$  has perimeter length 6, and each triangle touching  $T$  contains a piece of length at least  $1/2$  of  $R$ .  $\square$



## 79. Even and Odd Graphs

*The vertex set of every graph can be partitioned into two sets that induce even subgraphs. There is also a partition in which one of the sets induces an even subgraph and the other an odd subgraph.*

*Proof.* Let us prove the first assertion by induction on the number  $n$  of vertices of our graph  $G$ . For  $n = 1$  there is nothing to prove, so let us pass to the induction step. If  $G$  itself is even, the trivial partition  $V(G) = V(G) \cup \emptyset$  will do. Otherwise, let  $x \in V(G)$  be a vertex of odd degree, and let  $U$  be the set of vertices adjacent to  $x$ . Let  $G'$  be the graph obtained from  $G - x$  by toggling the pairs of vertices of  $U$ . Thus, two vertices of  $U$  are adjacent in  $G'$  if and only if they are not adjacent in  $G$ , and all other pairs of vertices are adjacent in  $G'$  if and only if they are adjacent in  $G$ . By the induction hypothesis,  $V(G')$  has a partition  $V(G') = V_1 \cup V_2$  such that the induced subgraphs  $G'[V_1]$  and  $G'[V_2]$  are even. Since  $|U|$  is odd, we may assume that  $|V_1 \cap U|$  is even and  $|V_2 \cap U|$  is odd. Then the sets  $V_1 \cup \{x\}$  and  $V_2$  partition  $V(G)$  and induce even subgraphs.

The second assertion is an immediate consequence of the first; in fact, the two assertions are trivially equivalent. Indeed, given a graph  $G$ , let  $G'$  be obtained from  $G$  by adding a vertex  $v$  and joining  $v$  to every vertex of  $G$ . If  $V(G')$  is partitioned into  $V_1$  and  $V_2$  with  $v \in V_1$ , then  $V_1$  induces an even subgraph of  $G'$  if and only if  $V_1 \setminus \{v\}$  induces an odd subgraph of  $G$ , and  $V_2$  induces the same subgraph in  $G'$  and  $G$ .  $\square$

**Notes.** Gallai never published his theorem we have just proved: it first appeared in print in Lovász's collection of problems and exercises.

That a graph need not have a partition into two odd sets is shown by any complete graph of odd order, say, the triangle. However, Caro conjectured that, for some constant  $c > 0$ , every connected graph of order  $n > 1$  contains a set of at least  $cn$  vertices that induce an odd subgraph. This conjecture is still open, but Scott proved that the conjecture is true with  $cn$  replaced by  $n/900 \log n$ .

L. Lovász, *Combinatorial Problems and Exercises* Second edition, North-Holland Publishing Co., (1993), 635 pp.

A.D. Scott, Large induced subgraphs with all degrees odd, *Combin. Prob. and Comput.* **1** (1992), 335–349.

## 80. Packing Squares: the Moon–Moser Theorem

*Every set of squares of total area 1 can be packed into a square of area 2. Furthermore, this assertion is false if 2 is replaced by a smaller constant.*

*Proof.* Let  $Q_0, Q_1, \dots$  be squares of side-lengths  $a_0 \geq a_1 \geq \dots$  and total area  $A = \sum_{i=0}^{\infty} a_i^2$ . Let  $b = \sqrt{2A}$ , so that our task is to pack the squares into a  $b \times b$  square. To this end, we shall pack the squares into the infinite strip

$$S = \{(x, y) : 0 \leq x \leq b, y \geq 0\},$$

and then show that our packing does not go above height  $b$ .

The packing algorithm we shall use is not even ‘greedy’, but ‘generous’. First, place the largest square,  $Q_0$ , of side-length  $a = a_0 \leq b$ , into the bottom left-hand corner of the strip. Use the right-hand side of  $Q_0$  to divide  $S$  into a strip  $S_0$  of base  $a$  containing  $Q_0$ , and the remaining strip  $S_1$  of base length  $b - a$ . Then pack the squares in decreasing order, placing  $Q_1$  next to  $Q_0$  on the bottom line, followed by  $Q_2$ , and so on. Let  $Q_{j_1}$  be the first square that does not fit on the bottom line, because it protrudes from the strip to the right, as in Figure 71. At this point, cut off the bottom part of  $S$  containing  $Q_0$ , and continue the packing into the remainder  $S'$  of  $S$ , made up of  $S'_0$  and  $S'_1$ , the remnants of  $S_0$  and  $S_1$ . To be precise, place this square  $Q_{j_1}$  on top of  $Q_0$ , into the bottom left-hand corner of  $S'_0$ . Continue with

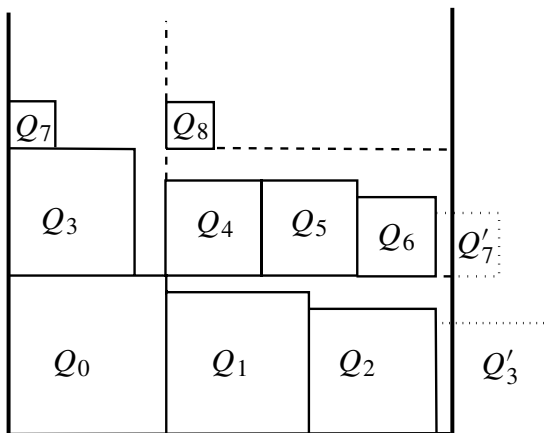


Fig. 71. The strip  $S$  and the ‘generous’ packing of the squares  $Q_0, Q_1, \dots, Q_8$ . The square  $Q'_3$  drawn with dotted lines is where we would like to place  $Q_3$ , but as there is not enough room there, we place it on top of  $Q_0$ . Similarly,  $Q_7$  is forced to go on top of  $Q_3$ .

$Q_{j_1+1}, Q_{j_1+2}, \dots$ , placing them on the bottom of  $S'_1$ , putting  $Q_{j_1+1}$  into the left-hand corner, then  $Q_{j_1+2}$  next to  $Q_{j_1+1}$  and touching it, and so on. Let  $Q_{j_2}$  be the first square that does not fit on the bottom of  $S'_1$ , because it protrudes to the right. Next, cut off the bottom of the strips up to the top of  $Q_{j_1}$  to obtain  $S''_0$  and  $S'_1$ ; place  $Q_{j_2}$  into the bottom left-hand corner of  $S''_0$  and the squares  $Q_{j_2+1}, Q_{j_2+2}, \dots$  on the bottom of  $S'_1$ , from the left-hand corner, say. The first square that does not fit,  $Q_{j_3}$ , goes again on the top of  $Q_{j_2}$ , into the bottom left corner of the next substrip,  $S'''_0$ , with the subsequent squares going into  $S'''_1$ , and so on.

This generous packing puts our squares into a  $b \times h$  rectangle, with  $h = a + \sum_{i=1}^{\infty} a_{j_i}$ , so all we have to check is that  $h \leq b$ . Note that

$$w_1 = \sum_{i=1}^{j_1} a_i \geq b - a,$$

since the squares  $Q_1, \dots, Q_{j_1}$  do not fit on the bottom of the strip  $S_1$  of base length  $b - a$ . Similarly,

$$w_2 = \sum_{j_1 < i \leq j_2} a_i \geq b - a,$$

and so on.

Consequently,

$$A - a^2 = \sum_{i=1}^{\infty} a_i^2 \geq \sum_{i=1}^{\infty} w_i a_{j_i} \geq (b - a) \sum_{i=1}^{\infty} a_{j_i},$$

so

$$h = a + \sum_{i=1}^{\infty} a_{j_i} \leq a + \frac{A - a^2}{b - a} = a + \frac{b^2/2 - a^2}{b - a} \leq b,$$

as claimed. Here the last inequality holds since it is equivalent to  $4a^2 - 4ab + b^2 \geq 0$ .

The constant 2 is clearly best possible: if  $\varepsilon > 0$  then two unit squares cannot be packed into a square of side-length  $2 - \varepsilon$ .  $\square$

**Notes.** The result above was proved by Moon and Moser, improving a less sharp result of Kosiński. Recently Januszewski extended the Moon–Moser result above: he showed that any family of rectangles, each of side-length at most 1, and total area at most  $1/2$ , can be packed into a unit square.

Fejes Tóth proposed the dual problem: given a compact convex subset  $C \subset \mathbb{R}^2$  with non-empty interior, what is  $f(C)$ , the infimum (really, minimum) of  $\alpha$ , such that if  $C_1, C_2, \dots$  are (positive) homothetic copies of  $C$  of total area  $\alpha$  times the area of  $C$ , then some translates of the  $C_i$  cover  $C$ ? Trivially,  $f(C) \geq 2$  for

every  $C$  (with non-empty interior). Moon and Moser proved that if  $Q$  is a square (equivalently, a parallelogram) then  $f(Q) = 3$ ; this is the dual of their result above. Bezdek and Bezdek noted that  $f(Q) = 3$  implies that  $f(D) \leq 12$  for a disc  $D$ ; this bound was improved by Januszewski to  $f(D) \leq 22/3$ . Bezdek and Bezdek also conjectured that the trivial lower bound  $f(C) \geq 2$  is attained if  $C$  is a triangle – this was proved by Füredi: the proof is considerably more difficult than that of the Moon–Moser result.

Concerning higher dimensions, Groemer, and Bezdek and Bezdek proved that if  $Q^d$  is a  $d$ -dimensional cube then  $f(Q^d) = 2^d - 1$ .

A. Bezdek and K. Bezdek, Eine hinreichende Bedingung für die Überdeckung des Einheitswürfels durch homothetische Exemplare im  $n$ -dimensionalen euklidischen Raum, *Beiträge Algebra Geom.* **17** (1984), 5–21.

L. Fejes Tóth, *Lagerungen in der Ebene auf der Kugel und im Raum*, Zweite verbesserte und erweiterte Auflage, Die Grundlehren der mathematischen Wissenschaften **65**, Springer-Verlag, (1972), xi + 238 pp.

L. Fejes Tóth, On the densest packing of convex discs, *Mathematika* **30** (1983), 1–3.

Z. Füredi, Covering a triangle with homothetic copies, in *Discrete Geometry*, a volume in honor of W. Kuperberg's 60th birthday, A. Bezdek (ed.), Monogr. Textbooks Pure Appl. Math. **253**, Dekker, (2003), pp. 435–445.

H. Groemer, Covering and packing properties of bounded sequences of convex sets, *Mathematika* **29** (1982), 18–31.

J. Januszewski, Translative covering of a convex body with its positive homothetic copies, in *Proceedings of the International Scientific Conference on Mathematics* (Žilina, 1998), Univ. Žilina, Žilina, (1998), pp. 29–34.

J. Januszewski, Packing rectangles into the unit square, *Geom. Dedicata* **81** (2000), 13–18.

A. Kosiński, A proof of an Auerbach–Banach–Mazur–Ulam theorem on convex bodies, *Colloq. Math.* **4** (1957), 216–218.

J.W. Moon and L. Moser, Some packing and covering theorems, *Colloq. Math.* **17** (1967), 103–110.

C.A. Rogers, *Packing and Covering*, Cambridge Tracts in Mathematics and Mathematical Physics **54**, Cambridge University Press, (1964), viii + 111 pp.

## 81. Filling a Matrix

Let  $A = (a_{ij})_{i,j=1}^{2k}$  be a matrix with entries  $1, \dots, 4k^2$ . Then some row or column contains two entries whose difference is at least  $2k^2 + k - 1$ . This bound is best possible.

*Proof.* As a warm-up, let us prove the assertion in the original problem that some two entries in the same row or column have difference at least  $2k^2$ . If the entries 1 and  $4k^2$  are in the same row or column then this is certainly the case. Otherwise, let  $a$  be the entry in the intersection of the row containing 1 and the column containing  $4k^2$ . Then the maximal difference in a row or column is at least

$$\max\{4k^2 - a, a - 1\} \geq \left\lceil \frac{1}{2} \left( (a - 1) + (4k^2 - a) \right) \right\rceil \geq 2k^2.$$

After this very simple proof of our warm-up, let us give another proof: this needs a little more work, but is easily boosted to a proof of the required better bound,  $2k^2 + k - 1$ .

Let  $S$  be the set of entries occupied by  $1, 2, \dots, k^2$ , and let  $T$  be the union of the columns and rows of these entries. (Thus  $(i, j) \in T$  if either row  $i$  or column  $j$  has an entry which is ‘small’, i.e., at most  $k^2$ .)

If  $T$  is the union of  $a$  rows and  $b$  columns then  $|T| = 2ka + 2kb - ab$  and  $ab \geq |S| = k^2$ . Consequently,

$$\begin{aligned} |T| &= 2ka + 2kb - ab = 2ka + b(2k - a) \geq 2ka + (k^2/a)(2k - a) \\ &= 2k(a + k^2/a) - k^2 \geq 3k^2, \end{aligned}$$

since  $a + k^2/a \geq 2k^2$ .

Let  $a_{ij}$  be the maximal entry in  $T$ , i.e., the maximal entry with  $(i, j) \in T$ . Then  $a_{ij} \geq 3k^2$ , and either row  $i$  or column  $j$  has an entry which is at most  $k^2$ . Hence, the difference of two such entries is at least  $2k^2$ , completing the proof of the warm-up.

This argument shows that in order to keep the maximal difference small we had better pack  $1, 2, \dots, k^2$  into one corner, and  $3k^2 + 1, 3k^2 + 2, \dots, 4k^2$  into the diametrically opposite corner. Thus we are led to the ‘book type’ arrangement in Figure 72, which shows that we cannot do better than  $2k^2 + k - 1$ .

Finally, let us turn to the main part of the problem, that of improving our warm-up bound  $2k^2$  to  $2k^2 + k - 1$ . To this end, we shall make use of the fact that either  $S$  is a  $k \times k$  submatrix of  $A$  or  $|T| \geq 3k^2 + k$ . In the first case the maximal entry  $a_{ij}$  with  $(i, j) \in T$  shares a row or column with  $k$  entries of  $S$ , so some difference is at least  $a_{ij} - (k^2 - k + 1) \geq 2k^2 + k - 1$ ; in the second case some difference is at least  $3k^2 + k - k^2 = 2k^2 + k$ .

36	40	44	48	52	56	60	64
35	39	43	47	51	55	59	63
34	38	42	46	50	54	58	62
33	37	41	45	49	53	57	61
4	8	12	16	20	24	28	32
3	7	11	15	19	23	27	31
2	6	10	14	18	22	26	30
1	5	9	13	17	21	25	29

Fig. 72. An arrangement of the numbers  $1, 2, \dots, 64$  in an  $8 \times 8$  matrix. Any two elements in the same row or column differ by at most  $2 \cdot 4^2 + 4 - 1 = 35$ .

To complete our proof, suppose that  $S$  is not a  $k \times k$  submatrix. Then, assuming, as we may, that  $a \geq b$ , we find that  $a = k + r \geq k + 1$  and  $b \geq k - r + 1$ , so

$$\begin{aligned}
 |T| &\geq 2k(2k + 1) - (k + r)(k - r + 1) = 4k^2 + 2k - (k^2 - r^2 + k - r) \\
 &= 3k^2 + k + r^2 - r \geq 3k^2 + k,
 \end{aligned}$$

as claimed. □

**Notes.** This problem is from József Balogh; see his paper with J. Csirik, Index assignment for two-channel quantization, *IEEE Trans. Inform. Theory* **50**, 2737–2751.

## 82. An Inequality Concerning Triangles: the Erdős–Mordell Theorem

Let  $O$  be a point within a triangle  $ABC$ , and let  $P$ ,  $Q$  and  $R$  be the feet of the perpendiculars upon the sides. Then

$$OA + OB + OC \geq 2(OP + OQ + OR). \quad (32)$$

Furthermore, equality holds if and only if  $ABC$  is an equilateral triangle and  $O$  is its centroid.

*First Proof.* Let  $P$ ,  $Q$  and  $R$  be the feet of the perpendiculars from  $O$  upon the sides  $BC$ ,  $CA$  and  $AB$ , and let  $P_1P_2$ ,  $Q_1Q_2$  and  $R_1R_2$  be the perpendicular projections of the segments  $RQ$ ,  $PR$  and  $QP$  on  $BC$ ,  $CA$  and  $AB$ , as in Figure 73.

Since  $P_1P + PP_2 \leq RQ$ ,  $Q_1Q + QQ_2 \leq PR$  and  $R_1R + RR_2 \leq QP$ , we have

$$OA + OB + OC \geq \frac{OA(P_1P + PP_2)}{RQ} + \frac{OB(Q_1Q + QQ_2)}{PR} + \frac{OC(R_1R + RR_2)}{QP}. \quad (33)$$

The quadrilateral  $BPOR$  is in the circle with diameter  $BO$ , so the angles  $BPR$  and  $BOR$  are equal. Therefore the triangles  $P_1PR$  and  $ROB$  are similar, and so

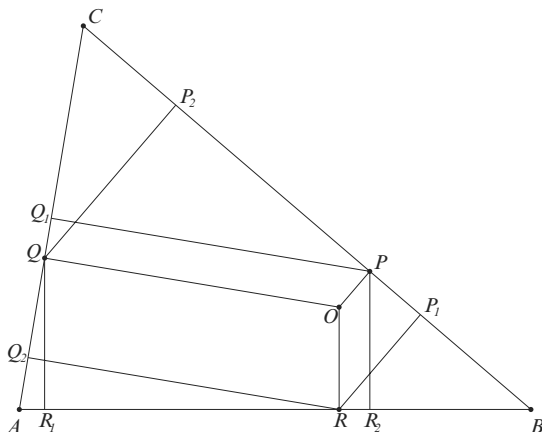


Fig. 73. The points  $P$ ,  $Q$  and  $R$  are the feet of the perpendiculars from  $O$  upon the sides  $BC$ ,  $CA$  and  $AB$ ;  $P_1$  and  $Q_2$  are the projections of  $R$ , etc.

$P_1P = PR \cdot OR/OB$ . Analogously,  $PP_2 = PQ \cdot OQ/OC$ ,  $Q_1Q = QP \cdot OP/OC$ ,  $QQ_2 = QR \cdot OR/OA$ ,  $R_1R = RQ \cdot OQ/OA$  and  $RR_2 = RP \cdot OP/OB$ . Substituting these expressions into (33) and grouping the terms with respect to the factors  $OP$ ,  $OQ$  and  $OR$ , we find that  $OA + OB + OC$  is no less than

$$\begin{aligned} & OP \left( \frac{RP \cdot OC}{PQ \cdot OB} + \frac{PQ \cdot OB}{RP \cdot OC} \right) + OQ \left( \frac{PQ \cdot QA}{RQ \cdot OC} + \frac{RQ \cdot OC}{PQ \cdot OA} \right) \\ & + OR \left( \frac{RP \cdot OA}{RQ \cdot OB} + \frac{RQ \cdot OB}{RP \cdot OA} \right) \geq 2(OP + OQ + OR), \end{aligned}$$

as claimed.

Now, let us turn to the case of equality. First, if  $ABC$  is equilateral and  $O$  is its centroid then we have equality in (32). Conversely, if we have equality in (32) then  $P_1P_2 = QR$  so  $QR$  is parallel to  $BC$ . Similarly,  $RP$  is parallel to  $CA$  and  $PQ$  to  $AB$ . This implies that  $P$ ,  $Q$  and  $R$  are the midpoints of their sides, and so  $ABC$  is an acute-angled triangle. (This is pretty obvious, but to spell it out note that if  $AR/AB = \lambda$  then  $AQ/AC = \lambda$ ,  $CQ/CA = CP/CB = 1 - \lambda$  and  $BP/BC = BR/BA = \lambda$ . Hence,  $AR/AB = BR/BA = \lambda$ , so  $R$  is the midpoint of  $AB$ . Similarly,  $P$  is the midpoint of  $BC$  and  $Q$  is the midpoint of  $CA$ .) Since  $P$ ,  $Q$  and  $R$  are the midpoints of their sides,  $OA = OB = OC$ . Without loss of generality, let us take  $OA = OB = OC = 1$ . Writing  $\alpha$ ,  $\beta$  and  $\gamma$  for the angles of the triangle  $ABC$ , we have

$$OP + OQ + OR = \cos \alpha + \cos \beta + \cos \gamma.$$

As the function  $\cos t$  is concave for  $0 \leq t \leq \pi/2$ , the maximum of this sum is attained only at  $\alpha = \beta = \gamma = \pi/3$ , i.e., when  $ABC$  is equilateral.  $\square$

*Second Proof.* Let  $A, B, C, O, P, Q$  and  $R$  be as in Figure 73. Set  $R_a = OA$  and  $r_a = OP$ , and define  $R_b, r_b, R_c$  and  $r_c$  analogously, so that inequality (32) is

$$R_a + R_b + R_c \geq 2(r_a + r_b + r_c).$$

Reflect  $O$  in the bisector of the angle  $BAC$  to obtain  $O'$ . Let  $\ell$  be the line through  $O'$  parallel to  $BC$ , and let  $B'$  be the intersection of  $\ell$  with  $AB$ . Let  $C'$  be defined analogously. Write  $a, b$  and  $c$  for the lengths of the sides of the triangle  $ABC$ , and  $a', b'$  and  $c'$  for the sides of  $AB'C'$ , so that  $a'/a = b'/b = c'/c$ . The triangles  $AOB'$  and  $AOC'$  have areas  $\frac{1}{2}c'r_b$  and  $\frac{1}{2}b'r_c$ , and the triangle  $ABC$  has area at most  $\frac{1}{2}a'R_a$ . Consequently,

$$a'R_a \geq c'r_b + b'r_c,$$

i.e.,

$$R_a \geq \frac{c}{a}r_b + \frac{b}{a}r_c,$$



with equality if and only if  $AO'$  is perpendicular to  $BC$ . Analogous inequalities hold for  $R_b$  and  $R_c$ . Summing these three inequalities, we find that

$$R_a + R_b + R_c \geq \left(\frac{b}{c} + \frac{c}{b}\right)r_a + \left(\frac{c}{a} + \frac{a}{c}\right)r_b + \left(\frac{a}{b} + \frac{b}{a}\right)r_c \geq 2(r_a + r_b + r_c),$$

with the second inequality an equality if and only if  $a = b = c$ . Hence inequality (32) does hold, and equality is attained if and only if the triangle  $ABC$  is equilateral and  $O$  is its centroid.  $\square$

**Notes.** This problem is a very old chestnut, which looks deceptively easy, but is unexpectedly difficult to solve (especially, in a nice way). I ‘tortured’ generations of Cambridge undergraduates with it without ever receiving a proper solution.

The problem is due to Paul Erdős, who first posed it in the Hungarian journal for secondary school students, and then as Problem 3739 in the *American Mathematical Monthly* in the June–July issue of 1935, p. 396. Upon hearing the problem, an eminent mathematician and close friend of Erdős threatened to give up mathematics if he failed to find a proof within a short space of time. In fact, he was unsuccessful in his attempt to find a proof but, fortunately for mathematics, he did not keep his promise and went on producing wonderful results. Eventually, a brief trigonometric proof was found by Erdős’ mentor, Mordell, which was published in 1935 in Hungarian (!), and then, in 1937, in the *Monthly*. Another solution, by Barrow, went a little further, and showed that equality holds if and only if  $ABC$  is an equilateral triangle and  $O$  is its centroid.

In 1953, in his book on discrete geometry, Fejes Tóth drew attention to the problem and the fact that there was no really simple proof. In answer to the appeal of Fejes Tóth, in 1957, D.K. Kazarinoff published a simple proof, and that simple proof was promptly extended to three dimensions by N.D. Kazarinoff. Since then, numerous other solutions and extensions have been given; for a selection of these results, see the references below.

In the first solution above we have reproduced the elegant and short proof of Bankoff; the second proof, which is, perhaps, even simpler, is due to Komornik.

In conclusion, let us mention one of the many extensions of the Erdős–Mordell inequality, the *weighted version* proved by Dar and Gueron.

Let  $\lambda_a$ ,  $\lambda_b$  and  $\lambda_c$  be positive numbers and, as in the original problem and in Figure 73, let  $O$  be a point of a triangle  $ABC$ , with  $P$  the foot of the perpendicular from  $O$  upon (the line determined by)  $BC$ ,  $Q$  the foot of the perpendicular upon  $CA$ , and  $R$  upon  $AB$ . Then

$$\lambda_a OA + \lambda_b OB + \lambda_c OC \geq 2(\sqrt{\lambda_b \lambda_c} OP + \sqrt{\lambda_a \lambda_c} OQ + \sqrt{\lambda_a \lambda_b} OR).$$

Furthermore, equality holds if and only if  $BC : CA : AB = \sqrt{\lambda_a} : \sqrt{\lambda_b} : \sqrt{\lambda_c}$ , and  $O$  is the circumcentre of the triangle  $ABC$ .

The case  $\lambda_a = \lambda_b = \lambda_c$  of this result is the original Erdős–Mordell inequality.

A. Avez, A short proof of a theorem of Erdős and Mordell, *Amer. Math. Monthly* **100** (1993), 60–62.

L. Bankoff, An elementary proof of the Erdős–Mordell theorem, *Amer. Math. Monthly* **65** (1958), 521.

H. Brabant, The inequality of Erdős–Mordell again, *Nieuw Tijdschr. Wisk.* **46** (1958/59), 87.

S. Dar and S. Gueron, A weighted Erdős–Mordell inequality, *Amer. Math. Monthly* **108** (2001), 165–168.

H.G. Eggleston, A triangle inequality, *Math. Gazette* **42** (1958), 54–55.

P. Erdős, Problem 3739, *Amer. Math. Monthly* **42** (1935), 396. Solutions by L.J. Mordell and D.F. Barrow, **44** (1937), 252–254.

L. Fejes Tóth, *Lagerungen in der Ebene, auf den Kugel und im Raum*, Springer-Verlag, (1953).

A. Florian, Zu einem Satz von P. Erdős, *Elem. Math* **13** (1958), 55–58.

D.K. Kazarinoff, A simple proof of the Erdős–Mordell inequality for triangles, *Michigan Math. J.* **4** (1957), 97–98.

N.D. Kazarinoff, D.K. Kazarinoff's inequality for tetrahedra, *Michigan Math. J.* **4** (1957), 99–104.

V. Komornik, A short proof of the Erdős–Mordell theorem, *Amer. Math. Monthly* **104** (1997), 57–60.

H. Lenhard, Verallgemeinerung und Verschärfung der Erdős–Mordellschen Ungleichung für Polygone, *Arch. Math.* **12** (1961), 311–314.

L.J. Mordell, Egy geometriai probléma megoldása (A solution of a problem in geometry; in Hungarian), *Középiskolai Matematikai és Fizikai Lapok (Mathematics and Physics Journal for Secondary Schools)* **11** (1935), 145–146.

L.J. Mordell, On geometric problems of Erdős and Oppenheim, *Math. Gaz.* **46** (1962), 213–215.

A. Oppenheim, The Erdős inequality and other inequalities for a triangle, *Amer. Math. Monthly* **68** (1961), 226–230.

I. Reiman, 60 éve jelent meg a Középiskolai Matematikai és Fizikai Lapokban az Erdős–Mordell tétel (The Erdős–Mordell theorem was published 60 years ago in the Mathematics and Physics Journal for Secondary Schools; in Hungarian), *Mathematics and Physics Journal for Secondary Schools* **45** (1995), 385–394.

R. Spira, The isogenic and Erdős–Mordell points of a simplex, *Amer. Math. Monthly* **78** (1971), 856–864.

J. Steinig, A comparison of two inequalities for the triangle, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 19–22.

G.R. Veldkamp, The Erdős–Mordell inequality, *Nieuw Tijdschr. Wisk.* **45** (1957/58), 193–196.

### 83. Perfect Difference Sets

Let  $p \geq 2$  be a prime and  $n = p^2 + p + 1$ . Then there are integers  $a_0 = 0 < a_1 < \dots < a_p < n$  such that for every  $b$ ,  $1 \leq b \leq n$  there are  $a_i$  and  $a_j$  such that either  $b = a_i - a_j$  or  $n - b = a_i - a_j$ .

*Proof.* The main problem is how to define a suitable sequence  $(a_i)_0^p$  – checking that it has the required properties is fairly routine. In order to define an appropriate sequence, we shall make use of the finite field  $\text{GF}(p^3)$  with  $p^3$  elements and prime field  $\mathbb{Z}_p$ .

Let  $g$  be a primitive element of  $\text{GF}(p^3)$ , so that

(i)  $g, g^2, \dots, g^{p^3-2}$  and  $g^{p^3-1} = g^{(p^2+p+1)(p-1)} = g^{n(p-1)} = 1$  are the  $p^3 - 1$  non-zero elements of  $\text{GF}(p^3)$ ,

(ii)  $h = g^n \in \mathbb{Z}_p$  (since  $h^{p-1} = 1$ ), and

(iii) if  $ag^2 + bg + c = 0$  for some  $a, b, c \in \mathbb{Z}_p$  then  $a = b = c = 0$ .

Set  $a_p = 0$ , and for  $m = 0, 1, \dots, p-1$ , i.e.,  $m \in \mathbb{Z}_p$ , define  $a_m$ ,  $0 < a_m < p^3 - 1$ , by

$$g^{a_m} = g + m.$$

We claim that all differences  $a_r - a_s$ ,  $r \neq s$ , are distinct modulo  $n$ ; this clearly suffices to complete the proof, since then the  $(p+1)p$  distinct differences  $a_r - a_s$  must take every value  $b$  exactly once modulo  $n$ , so for every  $0 < b < n$  there is a unique ordered pair  $(a_r, a_s)$  such that  $a_r - a_s \equiv b \pmod{n}$ .

Suppose that  $0 \leq r \neq s \leq p$  and  $0 \leq t \neq u \leq p$  are such that

$$a_r - a_s \equiv a_t - a_u \pmod{n},$$

i.e.,

$$a_r + a_u = a_s + a_t + kn$$

for some integer  $k$ . Our task is to show that  $r = t$  and  $s = u$ .

Let us consider four cases according to the number of zero terms among the indices  $r, s, t$  and  $u$ . First, suppose that all four indices  $r, s, t$  and  $u$  are non-zero. Then

$$(g+r)(g+u) = g^{a_r+a_u} = g^{a_s+a_t+kn} = (g+s)(g+t)v,$$

where  $v = g^{kn} = (g^n)^k = h^k \in \mathbb{Z}_p$ ,  $v \neq 0$ . Hence,  $g$  is a root of the polynomial  $v(g+s)(g+t) - (g+r)(g+u)$  of degree at most 2 over  $\mathbb{Z}_p$ ; consequently, this is the zero polynomial. Therefore,  $v = 1$ ,  $r = t$  (since  $r \neq s$ ) and  $s = u$ , as required.

The cases when one or two of the indices  $r, s, t$  and  $u$  are zero are even easier to handle. Indeed, arguing as above, if  $r = 0 < s \leq p$  and  $0 < t \neq u \leq p$  then

$$g + u = v(g + s)(g + t),$$

a contradiction, and if  $r = 0 < s \leq p$  and  $u = 0 < t \leq p$  then

$$v(g + s)(g + t) = 1,$$

which is also absurd.

Finally, if  $r = 0 < s \leq p$  and  $t = 0 < u \leq p$  (so that  $r = t$  by assumption), then  $a_u = a_s + kn$ , so  $g + u = v(g + s)$ , implying that  $v = 1$  and  $s = u$ .  $\square$

**Notes.** The result above was proved by Singer; its beautiful and simple proof reproduced above is due to Vijayaraghavan and Chowla.

J. Singer, A theorem in finite projective geometry and some applications to number theory, *Trans. Amer. Math. Soc* **43** (1938), 377-385.

T. Vijayaraghavan and S. Chowla, Short proofs of theorems of Bose and Singer, *Proc. Nat. Acad. Sci. India (A)* **15** (1945), 194.

## 84. Difference Bases

Let  $k(n)$  be the minimal number of elements in a difference basis of  $[n]$ . Then

$$\lim_{n \rightarrow \infty} k(n)/\sqrt{n} = c$$

for some constant  $c$ ,  $\sqrt{2} \leq c \leq (8/3)^{1/2}$ .

*Proof.* This assertion is considerably trickier than the usual run-of-the-mill limit questions based on subadditivity or submultiplicativity: here we have to be a little more careful.

Let

$$c = \inf_{n \geq 2} k(n)/\sqrt{n}.$$

Trivially,  $\sqrt{2} \leq c$ , since  $k(n)(k(n) - 1)/2 \geq n$ ; also,  $c \leq 4/\sqrt{6} = (8/3)^{1/2}$ , since  $k(6) \leq 4$ , as  $\{0, 1, 4, 6\}$  is a difference basis for 6. Our aim is to show that if  $n$  is large enough,  $k(n)$  is not much larger than  $c\sqrt{n}$ . To be precise,  $k(n)/\sqrt{n}$  does tend to  $c$  if, given  $\varepsilon > 0$ ,

$$k(N) < (1 + \varepsilon)c\sqrt{N},$$

provided  $N \geq N(\varepsilon)$ .

Let  $p_1 = 2, p_2 = 3, \dots$  be the sequence of primes, and set  $m_s = p_s^2 + p_s + 1$ . We know that  $p_s/p_{s-1} \rightarrow 1$  (and so  $m_s/m_{s-1} \rightarrow 1$ ) as  $s \rightarrow \infty$ . Let  $p_{s_0} > 1/\varepsilon$  be such that  $(m_s/m_{s-1})^{1/2} < 1 + \varepsilon$ . Also, let  $n \geq 2$  be such that

$$k(n) \leq (1 + \varepsilon)c\sqrt{n}.$$

We claim that  $N(\varepsilon) = m_{s_0}n$  will do.

Indeed, let  $s \geq s_0$  be such that

$$m_{s-1}n < N \leq m_s n.$$

We know from Problem 83 that for  $p = p_s$  and  $m = p^2 + p + 1$  there is a perfect difference set  $A \subset \mathbb{Z}_m$  for  $m$ , i.e., there are integers  $0 \leq a_0 < a_1 < \dots < a_p \leq m$  such that, given any  $a$ , with  $1 \leq a < m$ , then either  $a = a_i - a_j$  or  $a - m = a_i - a_j$  for some  $a_i$  and  $a_j$ .

Let  $0 \leq b_1 < \dots < b_k$  be a difference basis of  $[n]$  with  $k = k(n)$ . Then the set

$$\{a_i + mb_j : 0 \leq i \leq p, 1 \leq j \leq k\}$$

is a difference basis of  $[mn]$ , and so of  $[N]$ . Hence,

$$\begin{aligned} k(N) &\leq (p+1)k(n) \leq (1+\varepsilon)\sqrt{m}(1+\varepsilon)c\sqrt{n} \\ &< (1+\varepsilon)^2 c\sqrt{N}(m_s/m_{s-1})^{1/2} < (1+\varepsilon)^3 c\sqrt{N}, \end{aligned}$$

completing the proof.  $\square$

**Notes.** This result is due to Rédei and Rényi. In fact, rather unconventionally, the authors state in their paper that the existence of the limit was proved by Rédei, while Rényi gave the following proof of a non-trivial lower bound for the limit  $c$ .

Let  $k = k(n)$  and let  $\{b_1, \dots, b_k\}$  be a basis of  $[n]$ . Then, for every  $1 \leq |d| \leq n$ , there are  $b_r, b_s$  with  $d = b_r - b_s$ . Therefore,

$$2 \sum_{d=1}^n \cos dx + k^2 - 2n \geq \sum_{r,s=1}^k \cos(b_r - b_s)x = \left| \sum_{r=1}^k e^{ib_r x} \right|^2 \geq 0,$$

so

$$2 \sum_{d=1}^n \cos dx \geq 2n - k^2.$$

All that remains is to choose a value of  $x$  for which the left-hand side is small. Such a value is  $x = 3\pi/(2n+1)$ :

$$2 \sum_{d=1}^n \cos 3d\pi/(2n+1) < -\frac{2}{3\pi}(2n+1) - 1.$$

Consequently,

$$(2n+1)(1 + 2/3\pi) < k^2,$$

implying that

$$c = \lim_{n \rightarrow \infty} k(n)/\sqrt{n} \geq \left(2 + \frac{4}{3\pi}\right)^{1/2}.$$

For numerous related results, see the references below.

A. Brauer, On a problem in number theory and its application in electrical engineering, *J. Elisha Mitchell Sci. Soc.* **61** (1945), 55–66.

P. Erdős and I.S. Gál, On the representation of  $1, 2, \dots, N$  by differences, *Indagationes Math.* **10** (1949), 379–382.

M.J.E. Golay, Notes on the representation of  $1, 2, \dots, N$  by differences, *J. London Math. Soc.* (2) **4** (1972), 729–734.

C.B. Haselgrove and J. Leech, Note on restricted difference bases, *J. London Math. Soc.* **32** (1957), 228–231.

J. Leech, On the representation of  $1, 2, \dots, N$  by differences, *J. London Math. Soc.* **31** (1956), 160–169.

J.C.P. Miller, Difference bases: three problems in the additive theory of numbers, in *Computers in Number Theory*, A.O.L. Atkin and B.J. Birch, (eds), Academic Press, (1971), pp. 299–322.

L. Rédei, and A. Rényi, On the representation of the numbers  $1, 2, \dots, N$  by differences (in Russian), *Mat. Sbornik N. S.* **24 (66)** (1949), 385–389.

B. Wichmann, A note on restricted difference bases, *J. London Math. Soc.* **38** (1962), 465–466.



## 85. Satisfied Cricketers: the Hardy–Littlewood Maximal Theorem

Let  $\mathbf{b} = (b_i)_1^n$  be a real sequence and let  $\mathbf{b}^* = (b_i^*)_1^n$  be its decreasing rearrangement. (For example, the decreasing rearrangement of 3, -1, 0, 3, 2, 5, -1, 3 is 5, 3, 3, 3, 2, 0, -1, -1.) Let  $\mathbf{a} = (a_i)$  be the sequence of maximal averages, so that

$$a_i = a_i(\mathbf{b}) = \max_{j \leq i} \frac{b_j + b_{j+1} + \cdots + b_i}{i - j + 1},$$

and define  $\mathbf{a}^* = (a_i^*)$ ,  $a_i^* = a_i(\mathbf{b}^*)$ , analogously. If  $s$  is an increasing function then

$$\sum_{i=1}^n s(a_i) \leq \sum_{i=1}^n s(a_i^*).$$

*Proof.* Since  $(b_i^*)$  is decreasing, for each  $i$ , the maximal average of the sequence  $\mathbf{b}^*$  ending at  $i$  is obtained by taking the average of all the first  $i$  values; in particular,  $(a_i^*)$  is also a decreasing sequence.

Note that the assertion in the question follows if we show that, for every  $x$ , the number of  $i$  with  $a_i \geq x$  is at most the number of  $i$  with  $a_i^* \geq x$ . (In fact, trivially, the assertion holds for every increasing function  $s$  if and only if this condition is satisfied.) Equivalently, our aim is to show that the number of  $i$  for which the

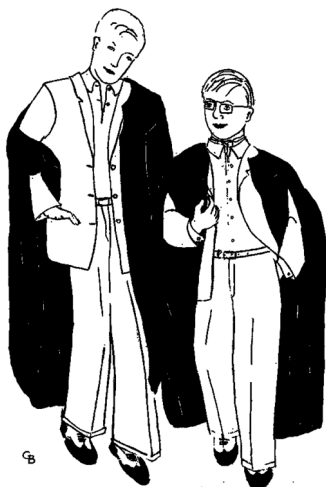


Fig. 74. Hardy and Littlewood.



average of the  $b_j$  over some interval ending at  $i$  is at least  $x$  is no more than the number of  $i$  with  $a_i^* \geq x$ .

Fix a real number  $x \leq \max b_i = b_1^*$ , and let  $k \leq n$  be the maximal integer such that  $a_k^* \geq x$ . If  $k = n$ , there is nothing to prove. Otherwise,

$$a_1^* = b_1 \geq \cdots \geq a_k^* = \frac{1}{k} \sum_{j=1}^k b_j^* \geq x > a_{k+1}^* = \frac{1}{k+1} \sum_{j=1}^{k+1} b_j^*;$$

in particular, the average of any  $k+1$  of the  $b_j$  is less than  $x$ .

Let  $I(b; x)$  be the set of indices  $i$  such that  $a_i \geq x$ , and for  $i \in I(b; x)$  let  $i'$  be the maximal integer such that

$$a_i \geq \frac{1}{i - i' + 1} \sum_{j=i'}^i b_j \geq x.$$

Our aim is to prove that  $|I(b; x)| \leq k$ .

Note that there are subtle but important differences between  $i^*$ , chosen in the question itself, and  $i'$ , defined above. Firstly,  $i'$  is maximal rather than minimal, and secondly and more importantly, the average of the  $b_j$  over the closed interval  $[i', i] = \{i', i' + 1, \dots, i\}$  is at least  $x$ , and not  $a_i$  (By definition,  $a_i \geq x$  for  $i \in I(b; x)$ .)

If  $i' \leq h < i$  for some  $h \in I(b; x)$  then  $[h', h] \subset [i', i]$  since otherwise the average of the  $b_j$  from  $h+1$  to  $i$  is less than  $x$ , and so is the average of the  $b_j$  from  $i'$  to  $h$ , implying the contradiction that the average of the  $b_j$  from  $i'$  to  $i$  is less than  $x$ . Hence, the set

$$J(b; x) = \bigcup_{i \in I(b; x)} [i', i],$$

which clearly contains  $I(b; x)$ , is the *disjoint* union of some intervals  $[i', i]$  with  $i \in I(b; x)$ . Since the average of the  $b_j$  over each  $[i', i]$  is at least  $x$ , we find that the average of the  $b_j$  over the entire set  $J(b; x)$  is also at least  $x$ . Therefore,

$$|I(b; x)| \leq |J(b; x)| \leq k,$$

completing the proof. □

**Notes.** The result above is the discrete form of the celebrated *maximal theorem* of Hardy and Littlewood; the proof we have presented is considerably simpler than the original proof from 1930. For other simple proofs, see Gabriel, Rado, Paley and Riesz.

Hardy, who loved cricket, explained the result in terms of the satisfaction of batsmen: in reverence to him, my formulation of the question is close to the original.

Hardy and Littlewood also reformulated their inequality in terms of averages over varying intervals as follows. Given a sequence  $b = (b_i)_1^n$ , for  $1 \leq h \leq k \leq n$  set

$$A(k, h) = A(k, h; b) = \frac{b_h + b_{h+1} + \cdots + b_k}{k - h + 1}.$$

If  $s(x)$  is an increasing function and  $h(k)$  is a function of  $k$  which is a positive integer for every positive integral  $k$ , and never exceeds  $k$ , then

$$\sum_{k=1}^n s(A(k, h(k), b)) \leq \sum_{k=1}^n s(A(k, 1, b^*)),$$

where  $b^*$  is the decreasing rearrangement of  $b$ .

The discrete form of the inequality we have presented trivially implies the maximal inequality for integrals Hardy and Littlewood needed for applications. To formulate this inequality, given a positive, bounded, and measurable function  $f(x)$  in  $[0, 1]$ , say, write  $m(y)$  for the measure of the set in which  $f(x) \geq y$ . Define  $f^*(x)$ , for  $0 \leq x \leq 1$ , by

$$f^*(m(y)) = y.$$

We call  $f^*(x)$  the *decreasing rearrangement of  $f(x)$* . It is evident that  $f$  and  $f^*$  are ‘equimeasurable’.

For  $0 \leq \xi < x \leq 1$ , write

$$A(x, \xi) = A(x, \xi, f) = \frac{1}{x - \xi} \int_{\xi}^x f(t) dt,$$

and set  $A(x, x) = f(x)$ . If  $f(x)$  is bounded,  $A(x, \xi)$  is bounded; it is clearly continuous in  $\xi$  except perhaps for  $\xi = x$ . Define  $\Theta(x)$  by

$$\Theta(x) = \Theta(x, f) = \sup_{0 \leq \xi \leq x} A(x, \xi).$$

When  $f(x)$  decreases,  $\Theta(x) = A(x, 0)$ .

Here is then the analogue (and immediate consequence) of the discrete maximal inequality in the Problem.

If  $s(x)$  is continuous and increasing, then

$$\int_0^1 s(\Theta(x, f)) dx \leq \int_0^1 s(\Theta(x, f^*)) dx.$$

Let us state one of the many important function-theoretic applications Hardy and Littlewood found of their maximal inequalities.

Let  $\lambda > 0$ . Let the complex function  $f(z)$  be regular for  $|z| < 1$  and such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^\lambda d\theta \leq C^\lambda$$

for  $0 < r < 1$ , where  $C$  is a positive constant. Put, for  $0 \leq \alpha < \pi/2$  and  $-\pi \leq \theta \leq \pi$ ,

$$F(\theta, \alpha) = \sup_{z \in S_\alpha(\theta)} |f(z)|,$$

where  $S_\alpha(\theta)$  denotes the interior of the convex quadrilateral having  $[0, e^{i\theta}]$  as diagonal and also as a line of symmetry and having angles  $2\alpha$  and  $\pi - 2\alpha$  at  $e^{i\theta}$  and 0, respectively. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F^\lambda(\theta, \alpha) d\theta \leq A(\lambda, \alpha) C^\lambda,$$

where  $A$  depends only on  $\lambda$  and  $\alpha$ .

R.M. Gabriel, An additional proof of a maximal theorem of Hardy and Littlewood, *J. London Math. Soc.* **6** (1931), 163–166.

G.H. Hardy and J.E. Littlewood, A maximal theorem with function-theoretic applications, *Acta Mathematica* **54** (1930), 81–116.

See also its reprint in the *Collected Papers of G.H. Hardy*, edited by a committee appointed by the London Mathematical Society, vol. II., Clarendon Press, (1967), pp. 509–545.

R.E.A.C. Paley, A proof of a theorem on averages, *Proc. London Math. Soc.* (2) **31** (1930), 289–300.

R. Rado, A new proof of a theorem of Hardy and Littlewood, *J. London Math. Soc.* **11** (1936), 87–92.

F. Riesz, Sur un théorème de maximum de MM. Hardy et Littlewood, *J. London Math. Soc.* **7** (1932), 10–13.

## 86. Random Words

Consider a random word of length  $2n$ , with  $n$  letters  $A$  and  $n$  letters  $B$ . For  $0 \leq m \leq n$ , let  $p(m, n)$  be the probability that there are precisely  $m$  letters  $B$  in the word that are preceded by at most as many  $A$ s as  $B$ s. Then  $p(m, n) = 1/(n+1)$  for every  $m$ .

*Proof.* Given an  $A$ - $B$  word (with  $n$  letters  $A$  and  $n$  letters  $B$ ), write  $b_0$  for the number of  $A$ s preceding the first  $B$  in the sequence,  $b_1$  for the number of  $A$ s between the first  $B$  and the second, and so on; finally, write  $b_n$  for the number of  $A$ s after the last  $B$ . Thus,  $\sum_{i=0}^n b_i = n$ . Set  $a_i = 1 - b_i$ , so that  $a_i \leq 1$  and  $\sum_{i=0}^n a_i = 1$ . Note that our words are in one-to-one correspondence with the  $n$ -term integer sequences  $(a_i)_0^n$  summing to 1, with each  $a_i$  at most 1. The number of  $B$ s of our word that are preceded by at most as many  $A$ s as  $B$ s is precisely the number of partial sums  $\sum_{i=0}^k a_i$  that are at most 0.

Let us associate with each sequence  $\mathbf{a} = (a_i)_0^n$  the  $n+1$  sequences obtained from it by cyclic permutations. (*A priori* it is not obvious that these  $n+1$  sequences are all different, but this will follow from the argument below.) Let us consider the suffices modulo  $n+1$ , and, as in Problem 7, write  $[k, \ell)$  for the set  $\{k, k+1, \dots, \ell-1\}$  consisting of at least one and at most  $n+1$  elements, so that  $[k, \ell) \cup [\ell, k) = [k, k) = \mathbb{Z}_{n+1}$  for all  $k \neq \ell$ . Furthermore, set  $S[k, \ell) = \sum_{i \in [k, \ell)} a_i$ , so that  $S[k, k) = \sum_{i=0}^n a_i = 1$  for every  $k$ . For  $0 \leq k \leq n+1$ , set  $S_k = S_k(\mathbf{a}) = \{\ell : S[k, \ell) \geq 1\}$ . Since  $k \in S_k$  for every  $k$ , the cardinalities  $|S_k|$  take at most  $n+1$  values. To prove our assertion, it will suffice to show that, for each  $m$ ,  $m = 0, \dots, n$ , precisely one of the cardinalities  $|S_k|$  is  $n+1-m$ ; equivalently, that the cardinalities  $|S_0|, |S_1|, \dots, |S_n|$  are all different.

In fact, we shall prove more: given  $k \neq \ell$ , either  $S_k$  is a proper subset of  $S_\ell$  or  $S_\ell$  is a proper subset of  $S_k$ . To see this, note first that either  $k \notin S_\ell$  or  $\ell \notin S_k$  since otherwise  $S[k, k) = S[k, \ell) + S[\ell, k) \geq 2$ , contradicting the fact that  $S[k, k) = \sum_{i=0}^n a_i = 1$ . Hence, we may suppose that  $k \notin S_\ell$ , i.e.,  $S[\ell, k) \leq 0$ . Since  $k \in S_k$ , we see that  $S_k \neq S_\ell$ .

Now we have come to the essence of the solution. Suppose, for a contradiction, that  $S_\ell$  is not a proper subset of  $S_k$ , i.e.,  $S_\ell \setminus S_k \neq \emptyset$ ; say,  $h \in S_\ell \setminus S_k$ . Let us recap what we know about  $h, k$  and  $\ell$ :

$$S[\ell, k) \leq 0, \quad S[k, h) \leq 0, \quad \text{and} \quad S[\ell, h) \geq 1. \quad (34)$$

Note that the ‘cycle’  $\mathbb{Z}_{n+1}$  is either the disjoint union of the segments  $[h, k)$ ,  $[k, \ell)$  and  $[\ell, h)$ , or else it is the disjoint union of  $[k, h)$ ,  $[h, \ell)$  and  $[\ell, k)$ . In the former

case, the segments  $[\ell, k)$  and  $[k, h)$  cover  $\mathbb{Z}_{n+1}$ , with  $[\ell, h)$  doubly and the rest singly, implying that

$$S[\ell, k) + S[k, h) = S[\ell, \ell) + S[\ell, h) = 1 + S[\ell, h) \geq 2,$$

contradicting (34). In the latter case,  $[\ell, h)$  is the disjoint union of  $[\ell, k)$  and  $[k, h)$ . By (34), this implies the contradiction that

$$1 \leq S[\ell, k) = S[\ell, h) + S[h, k) \leq 0,$$

completing the proof. □

**Notes.** The assertion is a classical result of Chung and Feller; the proof above was given by Montágh.

Chung, K. L., and Feller, W., On fluctuations in coin-tossing, *Proc. Nat. Acad. Sci. USA* **35**, (1949), 605–608.

Montágh, B., A simple proof and a generalization of an old result of Chung and Feller, *Discrete Math.* **87** (1991), 105–108.

## 87. Crossing a Chess Board

*On an  $n \times m$  chess board, either a rook can move from the first row to the last using only black squares, or a king can move from the first column to the last using only white squares, but not both.*

*Proof.* Let us prove the main assertion that either a rook can move from the first row to the last using only black squares, or a king can move from the first column to the last using only white squares.

Extend the  $n$  by  $m$  board  $B$  (with  $n$  rows and  $m$  columns) to an  $n + 2$  by  $m + 2$  board by adding to it a column on the left, a column on the right, a row at the bottom, and a row at the top. Colour the squares in the new columns white and those in the new rows and old columns black. Thus the new board  $B^*$  has  $nm + 2n + 2m + 4$  squares, with  $2m$  of the new squares black and  $2n + 4$  white. Finally, replace each square by the regular octagon inscribed in it, so that we have a tessellation with octagons and squares, with a square for each of the original inner vertices of the board. (Note that we do not use the original vertices of the board  $B$ .) Clearly, in the new board  $B^*$  every internal vertex is a vertex of two octagons and a square, so has degree 3. Let each octagon inherit the colour of its square, and leave the small squares (surrounding the original vertices) white. Finally, write  $a, b, c$  and  $d$  for the four ‘corner’ vertices between the black and white regions, and  $a', b', c'$  and  $d'$  for their neighbours, as in Figure 75.

Construct an oriented path  $P$  consisting of some of the edges of the octagons separating the black region from the white one as follows. Start at the top left corner  $a$ , keeping the black octagon on the left. Having crossed this edge  $aa'$ , there is precisely one way to continue the path through edges separating the black region from the white; also, as black is on the left of the first edge, the black region will stay on the left throughout the path. Continue in this way until we have to leave the new board  $B^*$ .

Since every edge of  $P$  has a black octagon on its left and a white region (an octagon or a square) on its right, the path  $P$ , starting with the edge  $\overrightarrow{aa'}$ , leaves  $B^*$  either at  $b$  (through the edge  $b'b$ ) or at  $d$  (through  $d'd$ ).

Now, if  $P$  is an  $a$ - $b$  path then the octagons on its left give us a top-to-bottom walk through neighbouring black octagons, and if it is an  $a$ - $d$  path then the white regions to the right of  $P$  give us a walk through neighbouring white regions from top to bottom from the left side to the right. In the former case, we have found a walk through black squares enabling a rook to cross the board  $B$  from top to bottom through black squares, and in the latter case we have found a walk through white

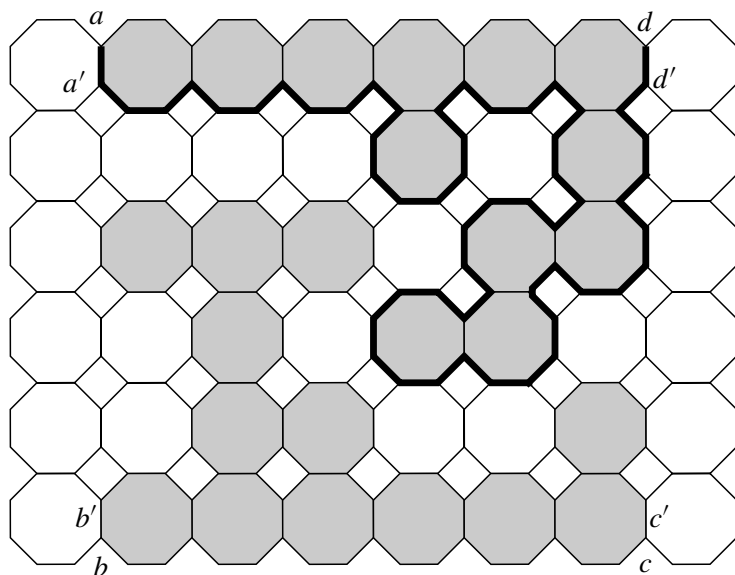


Fig. 75. A board  $B^*$  of octagons and squares constructed from a coloured chess board  $B$ , with a path from  $a$  to  $d$  between the black and white regions.

squares enabling a king to cross  $B$  from left to right using only white squares. This completes the proof of the main assertion.

That we cannot have a black top-to-bottom rook path *and* a white left-to-right king path follows instantly from the non-planarity of  $K_5$ , the complete 5-graph, which is a trivial consequence of Euler's formula.  $\square$

**Notes.** The above proof of this well known result is from B. Bollobás and O. Riordan, Sharp thresholds and percolation in the plane, *Random Structures and Algorithms*, to appear.

## 88. Powers of Paths and Cycles

- (i) *The maximal number of edges spanned by  $\ell$  vertices of the  $k$ th power of an infinite path is  $\ell(\ell - 1)/2$  for  $1 \leq \ell \leq k + 1$  and  $k\ell - k(k + 1)/2$  for  $\ell \geq k + 1$ .*
- (ii) *Let  $G$  be the  $k$ th power of an  $n$ -cycle, where  $3 \leq 2k + 1 \leq n$ . Then the minimal number of  $A$ - $B$  edges, subject to  $A$  and  $B$  partitioning the vertex set  $V(G)$  into sets with at least  $k$  vertices each, is  $k(k + 1)$ .*

*Proof.* (i) Let  $A = \{x_1, x_2, \dots, x_\ell\}$ , with  $x_1 < \dots < x_\ell$ , and  $A_0 = \{1, \dots, \ell\}$ . Note that if  $x_i < x_j \leq x_i + k$  then  $i < j \leq i + k$ . Hence  $A_0$  spans at least as many edges of  $G$  as  $A$ , so the maximal number of edges spanned by  $\ell$  vertices is as claimed.

(ii) If  $A$  is a  $k$ -arc of the  $n$ -cycle  $C_n$ , then in  $G = C_n^k$  there are  $k(k + 1)$  edges joining  $A$  to  $B = V(G) \setminus A$ . To prove the main assertion that there are at least this many  $A$ - $B$  edges, let us write  $a$  for the number of vertices in  $A$ ,  $b$  for the number of vertices in  $B$ ,  $r$  for the number of edges joining vertices of  $A$ ,  $s$  for the number of  $A$ - $B$  edges, and  $t$  for the number of edges joining vertices of  $B$ . Since  $G$  is  $2k$ -regular,  $2ka = 2r + s$ , and so our task is to show that  $r \leq ka - k(k + 1)/2$  or, equivalently, that  $t \leq kb - k(k + 1)/2$ . Now, this is obvious if  $a$  is  $k$  since  $r \leq k(k - 1)/2$  always holds: at best the vertices of  $A$  form a complete subgraph of  $G$ . By symmetry, the assertion is also obvious if  $a = n - k$ , since we always have  $t \leq b(b - 1)$  as well.

So far we have done essentially nothing: we have just tried to understand the problem. And now for the solution. Amazingly, we fix  $k$  and  $a$ , and apply induction on  $n$ . If  $n = a + k$  then  $b = k$  so  $t \leq k(k - 1)/2 = kb - k(k + 1)/2$ , as required. Suppose that  $b > k$  and the assertion holds for  $n - 1$ . Pick a vertex  $x$  of  $B$ , and let  $G'$  be obtained from  $G$  by deleting  $x$ . Clearly, we can add some (actually, precisely  $k$ ) edges to  $G'$  *not joining any two vertices of  $A$*  to turn  $G'$  into  $H = C_{n-1}^k$ , i.e., the ground graph with  $n - 1$  vertices. As this operation has not changed the edges joining vertices of  $A$ , by induction we have  $r \leq ka - k(k + 1)/2$ , completing the proof of the induction step and solving the problem.  $\square$

**Notes.** This very witty proof of part (ii), the main part of the question above, is from Paul Balister.



## 89. Powers of Oriented Cycles

Let  $n \geq 2k + 1 \geq 3$  and let  $\vec{G}$  be the standard orientation of  $C_n^k$ , the  $k$ th power of an  $n$ -cycle  $C_n$ . Then for every partition  $V(G) = A \cup B$  of the vertex set  $\mathbb{Z}_n$  into two sets with at least  $k$  vertices each, there are at least  $k(k + 1)/2$  edges going from a vertex in  $A$  to a vertex in  $B$ .

*Proof.* Let  $a$  be the number of vertices in  $A$  and  $b$  in  $B$ . If  $a$  is  $k$  (or  $k + 1$ ) then, as at most  $a(a - 1)$  of the  $ka$  edges originating in vertices in  $A$  also end in  $A$ , there are at least  $ka - a(a - 1)/2 = k(k + 1)/2$  edges oriented from  $A$  to  $B$ . By symmetry, the same holds if  $b$  is  $k$  (or  $k + 1$ ). In particular, the assertion holds for  $n = 2k + 1$ .

To prove the full assertion, we apply induction on  $n$ . Assume then that  $n > 2k + 1$  and the assertion holds for  $n - 1$ . If  $b$  is  $k$  (or  $k + 1$ ), the assertion is known to hold, so we may assume that  $b \geq k + 1$ . Pick a vertex  $i \in B$ , delete it from  $G$ , and add  $k$  directed edges to  $G - i$  so that the oriented graph obtained,  $H$ , say, is the  $k$ th power of an oriented  $(n - 1)$ -cycle. Let  $F$  be the set of edges from  $A$  to  $B$  in  $H$ . By the induction hypothesis,  $|F| \geq k(k + 1)/2$ . Note that for every vertex  $h \in A$ , there is at most one vertex  $\vec{j}$  such that  $\vec{hj} \in F$  but  $\vec{hj}$  is not an edge of  $G$ . Furthermore, if this happens then  $\vec{ij}$  is an edge of  $G$  from  $A$  to  $B$ . Consequently, in  $G$  there are at least  $|F| \geq k(k + 1)/2$  edges from  $A$  to  $B$ , completing the proof of the induction step and so the proof of the result.  $\square$

## 90. Perfect Trees

Suppose that there is a perfect weighted tree of order  $n$ . Then either  $n$  or  $n - 2$  is a perfect square.

*Proof.* Let  $T_w$  be a perfect weighted tree of order  $n$ . Fix a vertex  $x_0$  and set

$$V_0 = \{x \in V(T_w) : d_w(x, x_0) \text{ is even}\},$$

and

$$V_1 = \{x \in V(T_w) : d_w(x, x_0) \text{ is odd}\}.$$

Then  $d_w(x, y)$  is odd if and only if  $x$  and  $y$  belong to different classes: one to  $V_0$  and the other to  $V_1$ . Hence, if  $|V_0| = m$ , say, then there are  $m(n - m)$  odd distances. On the other hand, there are precisely  $\lfloor ((\binom{n}{2}) + 1)/2 \rfloor$  odd integers from 1 to  $\binom{n}{2}$ , so

$$m(n - m) = \lfloor ((\binom{n}{2}) + 1)/2 \rfloor$$

for some natural number  $m$ .

The rest of the solution is trivial: we have to check that this equation implies that either  $n$  or  $n - 2$  is a perfect square.

If  $n = 4k$  then with  $m = 2k - \ell$  we have  $(2k - \ell)(2k + \ell) = k(4k - 1)$ , and so  $k = \ell^2$  and  $n = 4\ell^2$ .

If  $n = 4k + 1$  then with  $m = 2k - \ell$  we have  $(2k - \ell)(2k + 1 + \ell) = k(4k + 1)$ , and so  $k = \ell^2 + \ell$  and  $n = (2\ell + 1)^2$ .

If  $n = 4k + 2$  then with  $m = 2k + 1 - \ell$  we have  $(2k + 1 - \ell)(2k + 1 + \ell) = 4k^2 + 3k + 1$ , so  $k = \ell^2$  and  $n = 4\ell^2 + 2$ .

Finally, if  $n = 4k + 3$  then with  $m = 2k + 1 - \ell$  we find that  $(2k + 1 - \ell)(2k + 2 + \ell) = 4k^2 + 5k + 2$ , and so  $k = \ell^2 + \ell$  and  $n = (2\ell + 1)^2 + 2$ .  $\square$

**Notes.** The problem of finding perfect trees was posed by J. Leech: Another tree labelling problem, *Amer. Math. Monthly* **82** (1975), 923–925. Leech also posed a number of variants of this problem. For example, given a tree  $T$ , let  $N(T)$  be the maximal integer such that for some weighting (labelling) with natural numbers all distances  $1, 2, \dots, N$  occur. Then  $k(n) = \max\{N(T) : T \text{ is a path on } n \text{ vertices}\}$  is precisely the function we studied in Problem 84.

More importantly, Leech asks what we can say about the function  $\ell(n) = \max\{N(T) : T \text{ is a tree on } n \text{ vertices}\}$ . He adds: “Should anyone think that these problems are abstrusely impractical, I remark that they seem to have originated in

the context of electrical networks in which the nodes are terminals and the lengths and distances are the electrical resistances between pairs of nodes.”

The result above is from H. Taylor: Odd path sums in an edge-labelled tree, *Mathematics Magazine* **50** (1977), 258–259.



## 91. Circular Sequences

Let  $X = (x_1, x_2, \dots, x_n)$  be a circular sequence of integers with sum  $S = \sum_{i=1}^n x_i$ . Then every reflection sequence starting with  $X$  is finite. Even more, if a reflection sequence starts with  $X$  and ends in the stable sequence  $X_\ell$  then both  $\ell$  and  $X_\ell$  depend only on  $X$  and not on the particular sequence chosen.

*Proof.* We say that a doubly infinite integer sequence  $\Sigma = (\dots, s_{-1}, s_0, s_1, \dots)$  is an  $n$ -lift if there is an integer  $S$  such that  $s_{i+n} = s_i + S$  for every  $i$ . Every  $n$ -lift  $\Sigma$  defines a circular sequence  $X(\Sigma) = (s_1 - s_0, s_2 - s_1, \dots, s_n - s_{n-1})$ . Observe that  $s_k < s_{k-1}$  if and only if the  $k$ th entry of  $X$  is negative. In particular,  $\Sigma$  is monotone increasing (in the sense that  $s_i \leq s_{i+1}$  for every  $i$ ) if and only if  $X(\Sigma)$  is stable. Also, if  $s_k < s_{k-1}$  and  $r_k(\Sigma)$ , the reflection of  $\Sigma$  in  $k$ , is obtained from  $\Sigma$  by swapping  $s_{rn+k}$  and  $s_{rn+k-1}$  for every  $r$ , then  $X(r_k(\Sigma)) = r_k(X(\Sigma))$ . Indeed, writing  $x_i = s_i - s_{i-1}$ , we find that  $s_k - s_{k-2} = (s_{k-1} - s_{k-2}) + (s_k - s_{k-1}) = x_{k-1} + x_k$ ,  $s_{k-1} - s_k = -(s_k - s_{k-1}) = -x_k$  and  $s_{k+1} - s_{k-1} = (s_{k+1} - s_k) + (s_k - s_{k-1}) = x_{k+1} + x_k$ .

After these observations, there is very little to do. Given  $X = (x_1, x_2, \dots, x_n)$ , let  $\Sigma_0(X) = (\dots, s_{-1}, s_0, s_1, \dots)$  be the canonical  $n$ -lift of  $X$ : for  $1 \leq i \leq n$  set  $s_i = \sum_{j=1}^i x_j$  and extend this sequence by setting  $s_{rn+i} = rS + s_i$  for  $r \in \mathbb{Z}$ , so that  $s_0 = 0$  and  $X(\Sigma) = X$ . (In fact, instead of  $\Sigma_0$  we could choose any sequence  $\Sigma$  with  $X(\Sigma) = X$ .) For  $1 \leq i \leq n$ , let  $\ell_i = \ell_i(\Sigma_0) = \ell_i(X)$  be the number of indices  $j > i$  such that  $s_i > s_j$ , and set  $\ell = \ell(\Sigma_0) = \ell(X) = \sum_{i=1}^n \ell_i$ . Since

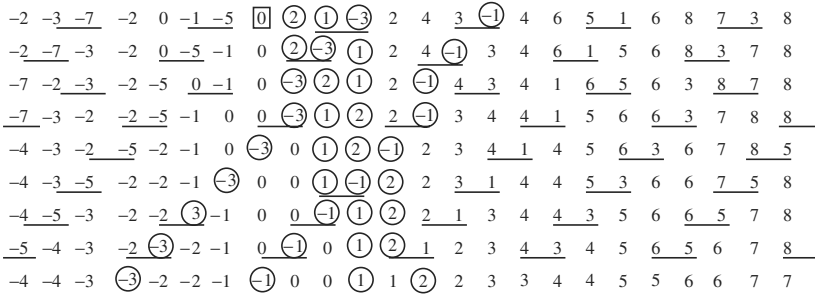


Fig. 76. A reflection sequence for  $(2, -1, -4, 5)$ , lifted to a doubly infinite sequence. We show the progression of the circled terms, with the terms to be swapped underlined. Note that the circled  $-3$  and  $-1$  are the same term modulo 4, and so they move in exactly the same way. The fourth route, that of the boxed 0 in the first row, which is the 0th term of the sequence is not shown. Note that this is also the route of the plain 2 in the first row. The final doubly infinite sequence is monotone increasing.

$S > 0$ , every  $\ell_i$  is finite, and so so is  $\ell$ . We claim that every reflection sequence  $X_0 = X, X_1, \dots$  has exactly  $\ell + 1$  terms, i.e., ends in  $X_\ell$ .

Indeed, every reflection sequence  $X_0, X_1, \dots$  of circular sequences corresponds to a reflection sequence  $\Sigma_0, \Sigma_1, \dots$  of  $n$ -lifts such that if  $X_{i+1} = r_k(X_i)$  then  $\Sigma_{i+1} = r_k(\Sigma_i)$ . Now, a reflection sequence of  $n$ -lifts gets sorted in a trivial ‘bubble sort’ manner:  $\Sigma_{i+1}$  is obtained from  $\Sigma_i$  by swapping two neighbouring terms  $s_{k-1}$  and  $s_k$  that are ‘the wrong way’, i.e., such that  $x_{k-1} > x_k$ . Needless to say, when we swap  $x_{k-1}$  and  $x_k$ , we have to swap  $s_{rn+k-1}$  and  $s_{rn+k}$  for every integer  $r$ . Swapping two neighbouring terms decreases  $\ell$  by 1:  $\ell(\Sigma_{i+1}) = \ell(\Sigma_i) - 1$ , hence, the sequence  $\Sigma_0, \Sigma_1, \dots$  indeed ends in  $\Sigma_\ell$ , when  $\ell(\Sigma_\ell) = 0$ .

Where will a term  $s_i$  starting in position  $i$  end up in the final sequence  $\Sigma_\ell$ ? It has to be interchanged with each of the  $\ell_i$  terms  $s_j$  such that  $i < j$  and  $s_i > s_j$ , and also with each of the  $m_i$  terms  $s_h$  such that  $h < i$  and  $s_h > s_i$ . Moreover,  $s_i$  will not be interchanged with any other term. But then  $s_i$  is moved to the right  $\ell_i$  times and to the left  $m_i$  times, so it ends up in position  $i + \ell_i - m_i$ , *independently* of the particular sequence chosen. But then  $\Sigma_\ell$  is independent of the route chosen, and therefore so is  $X_\ell$ .  $\square$

**Notes.** The problem was proposed as a coffee time problem by Anthony Quas, and the solution above was suggested right away by Alexander Scott. The question originated in a problem at the 1986 Mathematical Olympiad, and the extension above was proved by N. Alon, I. Krasikov and Y. Peres: Reflection sequences, *Amer. Math. Monthly* **96** (1989), 820–823, who proved several other results as well.

## 92. Infinite Sets with Integral Distances

*The plane does not contain an infinite set of non-collinear points with all pairs of points at integral distances.*

*Proof.* We shall prove the stronger assertion given in the *Hint*.

Let  $A, B, C$  be three non-collinear points in the plane. Then there are at most  $4(k+1)^2$  points  $P$  such that  $PA - PB$  and  $PB - PC$  are integral, where  $k = \lfloor \max(AB, BC) \rfloor$ .

Indeed, let  $P$  be a point with  $PA - PB$  and  $PB - PC$  integral. Since  $|PA - PB|$  is at most  $AB$ , it assumes one of the values  $0, 1, \dots, k$ , and so  $P$  lies on one of  $k+1$  hyperbolas. Similarly,  $P$  lies on one of the  $k+1$  hyperbolas determined by  $B$  and  $C$ . These (distinct) hyperbolas intersect in at most  $4(k+1)^2$  points.  $\square$

**Notes.** The result above is due to Anning and Erdős, but the proof we have presented was given by Erdős in a separate later paper. In his review of that, Kaplansky quoted the *entire paper*. It is amazing that although this is one of the ultimate *book proofs*, in a collection of papers of Erdős the original one of Anning and Erdős is reproduced, but not the brilliant brief follow-up.

Erdős notes also that an analogous theorem holds for higher dimensions.

N.H. Anning and P. Erdős, Integral distances, *Bull. Amer. Math. Soc.* **51** (1945), 598–600.

P. Erdős, Integral distances, *Bull. Amer. Math. Soc.* **51** (1945), 996.

P. Erdős, *The Art of Counting: Selected Writings*, J. Spencer (ed), with a dedication by R. Rado, Mathematicians of Our Time, The MIT Press (1973), xxiii + 742 pp.

I. Kaplansky, *Mathematical Reviews* **7** (1946), 164.

### 93. Finite Sets with Integral Distances

*For every  $n \geq 3$  there are  $n$  points in the plane with no three of them on a line such that the distance between any two is an integer.*

*Proof.* We shall prove more: *there are infinitely many points in the plane, no three on a line, such that the distance between any two is rational.* This is indeed more than what we have to show since a suitable multiple of  $n$  of these points gives us  $n$  points with integral distances.

To find a suitable infinite sequence of points with rational distances, we start with distinct primitive Pythagorean triples  $(a_i, b_i, c_i)$ ,  $1 < a_i < b_i < c_i$ ,  $i = 1, 2, \dots$ . (Thus  $a_i^2 + b_i^2 = c_i^2$  and  $a_i, b_i$  and  $c_i$  are relatively prime.) For  $i = 1, 2, \dots$ , let  $(x_i, y_i)$ ,  $x_i, y_i > 0$ , be the point of the circle  $x^2 + y^2 = 1/4$  of radius  $1/2$  whose distance from  $(-1/2, 0)$  is  $a_i/c_i$ . Then the distance of this point  $(x_i, y_i)$  from  $(1/2, 0)$  is  $b_i/c_i$  since  $(a_i/c_i)^2 + (b_i/c_i)^2 = 1$ . But then, for  $i \neq j$ , five of the six distances among the four concyclic points  $(-1/2, 0)$ ,  $(1/2, 0)$ ,  $(x_i, y_i)$ ,  $(x_j, y_j)$  are clearly rational, and therefore, by Ptolemy's theorem, so is the sixth, the distance between  $(x_i, y_i)$  and  $(x_j, y_j)$ . Hence the distance between any two points of the infinite sequence  $(x_1, y_1), (x_2, y_2), \dots$  is rational, completing the proof.  $\square$

**Notes.** This was the 'easy' part of the paper of Erdős and Anning mentioned in Problem 92. After Erdős' brilliant proof presented there, one may be justified in considering this assertion to be the 'hard' part of that paper.

In fact, rather than taking an infinite sequence of Pythagorean triples (and so appealing to the trivial fact that if  $1 < v < u$  are relatively prime with one odd and the other even then  $(u^2 - v^2, 2uv, u^2 + v^2)$  is a primitive Pythagorean triple), Erdős and Anning used the (simple but non-trivial) result that if  $p_i$  is a prime congruent to 1 modulo 4 then  $p_i^2 = a_i^2 + b_i^2$  for some natural numbers  $a_i$  and  $b_i$ .

Let us remark that it is even easier to find infinitely many non-collinear points with all distances rational. Indeed, let  $((a_i, b_i, c_i))_{i=1}^{\infty}$  be the sequence of primitive Pythagorean triples, and take the points  $(0, 1)$ ,  $(a_1/b_1, 0)$ ,  $(a_2/b_2)$ ,  $(a_3/b_3), \dots$ . Since the distance between  $(0, 1)$  and  $(a_i/b_i, 0)$  is  $(1 + a_i^2/b_i^2)^{1/2} = ((a_i^2 + b_i^2)/b_i^2)^{1/2} = c_i/b_i$ , all distances are rational.

N.H. Anning and P. Erdős, Integral distances, *Bull. Amer. Math. Soc.* **51** (1945), 598–600.

## 94. Cube-free Words: Thue's Theorem

*There is an infinite word on a two-letter alphabet in which no block occurs three times in succession*

*Proof.* We shall show that the very simple substitutions  $w_0 = 01$  and  $w_1 = 01$  will do. Set  $u_0 = 0$ , and for  $k \geq 0$  let  $u_{k+1}$  be obtained from  $u_k$  by replacing each occurrence of 0 by the block 01, and each occurrence of 1 by 10. Thus  $u_0 = 0$ ,  $u_1 = 01$ ,  $u_2 = 01\,01$ ,  $u_3 = 01\,10\,10\,01$ ,  $u_4 = 01\,10\,10\,01\,10\,01\,01\,10$ , and so on. Equivalently,  $u_{k+1} = u_k \overline{u_k}$ , where  $\overline{v}$  is the word  $v$  in which every letter is *toggled*, i.e., 0 is replaced by 1, and 1 by 0. We call the  $2^k$  blocks of length 2 making up  $u_{k+1}$  the *syllables* of  $u_{k+1}$ . Thus, if  $u_{k+1} = a_1 a_2 \dots a_\ell$ , where  $\ell = 2^{k+1}$ , then each block  $a_{2i-1} a_{2i}$  is a syllable of  $u_{k+1}$ , and  $u_k = a_1 a_3 \dots a_{\ell-1}$ . Note that each syllable is either 01 or 10; in particular, each syllable has the same number of 0s and 1s (just one, as it happens).

Now, to turn to the property in our assertion, a *cube* is a word of the form  $uuu$ , where  $u$  is non-empty. A word is *cube-free* if it contains no block which is a cube, i.e., does not contain three (non-empty) identical consecutive blocks.

We claim that every  $u_k$  is cube-free. Since  $u_k$  is the initial segment of length  $2^k$  of  $u_{k+1}$ , there is an infinite word  $u = a_1 a_2 \dots$  such that  $a_j$  is the  $j$ th letter of  $u_k$  whenever  $j \leq 2^k$ . Hence, if  $u_k$  is cube-free then  $u$  is cube-free as well, and our assertion is proved.

Suppose for a contradiction that  $u_k$  contains a cube  $www$ , where  $w$  is a word of length  $\ell$ , say. We shall show that then  $u_{k-1}$  also contains a cube: as  $u_0$  is trivially cube-free, this will complete our proof. Note first that  $\ell$  is even. Indeed, otherwise either the first and second occurrences of  $w$  (in the block  $www$ ) meet the same syllable, or the second and third occurrences do. But then  $u_k$  contains two consecutive blocks  $w$  such that the first is made up of  $(\ell - 1)/2$  full syllables and the first letter of a syllable  $s$  (in the middle), and the second is made up of the second letter of the same syllable  $s$  and  $(\ell - 1)/2$  full syllables. Consequently, these two blocks do not contain the same number of 1s, let alone are the same! Indeed, if  $s = 01$  then 1 occurs  $(\ell - 1)/2$  times in the first block and  $(\ell + 1)/2$  times in the second, and if  $s = 10$  then these numbers are reversed. Hence,  $\ell$  is indeed even.

Next, note that we may assume that the  $w$ s are made up of full syllables. Indeed, otherwise translate each  $w$  by one forward or backward (but all in the same direction) to obtain blocks  $w_1$ ,  $w_2$  and  $w_3$  that are made up of full syllables. Since each syllable is determined by either of its digits (and the position of that digit, first or



second), we have  $w_1 = w_2 = w_3 = \tilde{w}$ , so that  $u_k$  contains the cube  $\tilde{w}\tilde{w}\tilde{w}$ , with each  $\tilde{w}$  made up of  $\ell/2$  full syllables.

Finally, given a cube  $www$  of  $u_k$ , with each  $w$  the union of full syllables, the leading (first) letters of these syllables form a cube  $w'w'w'$  in  $u_{k-1}$ , completing our proof.  $\square$

**Notes.** The author of this classical result, Thue, was the first to study avoidable configurations in infinite words. For numerous related results, see the collection *Lothaire*. In Problem 95 we continue this topic.

M. Lothaire, *Combinatorics on Words*, A collective work by Dominique Perrin, Jean Berstel, Christian Choffrut, Robert Cori, Dominique Foata, Jean Eric Pin, Guiseppe Pirillo, Christophe Reutenauer, Marcel-P. Schützenberger, Jacques Sakarovitch and Imre Simon, with a foreword by Roger Lyndon, edited and with a preface by Dominique Perrin, *Encyclopedia of Mathematics and its Applications*, **17**, Addison-Wesley, (1983), xix+238 pp. See also M. Lothaire, *Combinatorics on Words*; corrected reprint of the 1983 original, with a new preface by Dominique Perrin, *Cambridge Mathematical Library*, Cambridge University Press, (1997), xviii + 238 pp.

A. Thue, Über unendliche Zeichenreihen, *Norske Vid. Selsk. Skr. I. Mat. Nat. Kl.*, Christiania, (1906), no. 7, 1–22.

A. Thue, Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen, *Norske Vid. Selsk. Skr. I. Mat. Nat. Kl.*, Christiania, (1912), no. 1, 1–67.

## 95. Square-free Words: the Thue–Morse Theorem

*There an infinite word on the alphabet 1, 2, 3 in which no block occurs twice in succession.*

*Proof.* We shall construct a sequence of square-free words  $u_0, u_1, \dots$  with  $u_n$  having length  $|u_n| = 3^n$ . To this end, we set  $u_0 = 1$  and having constructed  $u_n$ , we define  $v_n$  to be the *decorated* word obtained from  $u_n$  by replacing each digit  $i$  that is in an even place by  $\bar{i}$ . Thus  $v_n$  is a word on the six-letter alphabet 1,  $\bar{1}$ , 2,  $\bar{2}$ , 3 and  $\bar{3}$ . Given this word  $v_n$ , perform the following substitutions to obtain  $u_{n+1}$ : replace 1 by 123,  $\bar{1}$  by 321, 2 by 231,  $\bar{2}$  by 132, 3 by 312 and  $\bar{3}$  by 213. We may wish to emphasize the origin of the letters in  $u_{n+1}$ : in that case we use parentheses to indicate the *links* in  $u_{n+1}$ , the blocks coming from individual digits of  $v_n$ . Thus  $v_0 = 1$ ,  $u_1 = 123$ ,  $v_1 = 1\bar{2}3$ ,  $u_2 = (123)(132)(312) = 123\ 132\ 312$ ,  $v_2 = 1\bar{2}3\ \bar{1}\bar{3}\bar{2}\ 3\bar{1}\bar{2}$ ,

$$\begin{aligned} u_3 &= (123)(132)(312)(321)(312)(132)(312)(321)(231) \\ &= 123\ 132\ 312\ 321\ 312\ 132\ 312\ 321\ 231, \end{aligned}$$

and so on.

We shall prove that each  $u_n$  is square-free: this will suffice to complete the proof. Indeed,  $u_n$  has length  $|u_n| = 3^n$  and is the initial block of length  $3^n$  of  $u_{n+1}$ , so there is an infinite word  $u$  which is the ‘limit’ of the words  $u_0, u_1, u_2, \dots$ : the  $k$ th letter of  $u$  is the  $k$ th letter of  $u_n$  whenever  $3^n \geq k$ . Hence, if each  $u_n$  is square-free then so is  $u$ , as required. (In fact, looking for an infinite word as the ‘limit’ of finite words is a bit of a red herring: an easy compactness argument tells us that if there are arbitrarily long square-free words on a finite alphabet then there is an infinite square-free word on the same alphabet.)

Suppose for contradiction that not every  $u_n$  is square-free. Let  $n$  be the minimal integer such that  $u_n$  contains a square and let  $ww$  be a shortest square in  $u_n$ . Note that ‘increasing’ links (123, 231 and 312) and ‘decreasing’ ones (321, 132 and 213) alternate in every word  $u_m$ . This easily implies that  $|w| \geq 3$ . For example, if  $u_n$  contained 1212 then it would contain the successive increasing links (312) and (123). Also, if  $u_n$  contained 11 then it would contain either (321)(123) or (231)(132), and so  $v_{n-1}$  would contain  $\bar{1}\bar{1}$  or  $2\bar{2}$ , and so  $u_{n-1}$  would contain either 11 or 22, contradicting the minimality of  $n$ .

Let us embark on the essential part of our proof. In order to be able to refer to the two halves of the square  $ww$ , we write this occurrence of  $ww$  as  $ww'$  or  $w|w'$ .

We shall consider two cases according to the position of the ‘break’  $|$  between  $w$  and  $w'$ , and three subcases according to the length  $|w| = |w'|$  modulo 3.

**Case 1.** The break in  $ww'$  is ‘clean’: the last digit of  $w$  is the last digit of its link (and so the first digit of  $w'$  is the first digit of its link). To make it easier to follow the arguments, let us assume, as we may, that the first link of  $w'$  is (123), so that  $w$  also starts with 123.

**Subcase 1a:**  $|w| = 3k$ , i.e.,  $w$  consists of  $k$  full links, and so does  $w'$ . Then, as each link determines the decorated digit of  $v_{n-1}$  it comes from,  $v_{n-1}$  contains a square as well; *a fortiori*,  $u_{n-1}$  also contains a square, contradicting the minimality of  $n$ .

**Subcase 1b:**  $|w| = 3k + 1$ . Then  $w = 1)(231) \dots (\dots)$ , so 1 occurs  $k + 1$  times in  $w$ , and so in  $w'$  as well. Hence the last digit of  $w'$  is 1. But then, as  $w = w'$ , the last digit of  $w$  is also 1. This gives us the contradiction that  $u_n$  contains the square 11 (made up of the last digit of  $w$  and the first digit of  $w'$ ).

**Subcase 1c:**  $|w| = 3k + 2$ , say,  $w$  is the concatenation of 12 and  $k$  full links:

$$ww' = 12)(321) \dots (\dots)|(123) \dots (\dots)(\dots)$$

Now, since 1 and 2 occur  $k + 1$  times in  $w$ , the last two digits of  $w'$  are 1 and 2 in some order; therefore the last two digits of  $w$  are also 1 and 2. Consequently, the block of length four at the centre of  $ww'$  is either 12)(12 or 21)(12, and so  $u_n$  contains either the square 1212 or the square 11 – a contradiction.

**Case 2.** The break in  $ww'$  is ‘bad’, i.e., it cuts across a link. By symmetry, we may assume that  $ww' = \dots (1|23) \dots$

**Subcase 2a:**  $|w| = 3k$ . Then

$$ww' = 23)(\dots) \dots (\dots)(1|23)(\dots) \dots (\dots)(1$$

Consequently, by shifting  $ww'$  by one place to the left, with  $w$  going into  $w_0$  and  $w'$  into  $w'_0$ , we find that  $w_0 = w'_0$ , so  $u_n$  contains the square

$$w_0w'_0 = (123) \dots (\dots) | (123) \dots (\dots)$$

with the blocks  $w_0, w'_0$  made up of full links. By Subcase 1a, we know that this is impossible.

**Subcase 2b:**  $|w| = 3k + 1$ , so that

$$ww' = (231) \dots (1|23) \dots (\dots)(\dots)1$$

But then 1 occurs  $k + 1$  times in  $w$  and only  $k$  times in  $w'$ , contradicting the fact that  $w = w'$ .

**Subcase 2c:**  $|w| = 3k + 2$ , so that

$$ww' = 2)(3\dots) \dots (\dots)(1|23)(\dots) \dots (\dots)1$$

Then 1 occurs  $k + 1$  times in  $w$  and  $k$  times in  $w'$ , implying again the contradiction that  $w \neq w'$ .  $\square$

**Notes.** Another substitution that works and is easy to check replaces 1 by 12312, 2 by 131232, and 3 by 1323132.

As we remarked at the end of the solution of Problem 94, Thue was the first to study avoidable substructures of infinite words: in particular, he was the first to prove the result in this question. Thue's result was rediscovered by Morse (see also Morse and Hedlund). Square-free words have been studied by a good many mathematicians: see the notes in the collection *Lothaire*.

An extension of square-freeness is Abelian square-freeness. An *Abelian square* is a non-empty word of the form  $uu'$  such that every letter occurs precisely the same number of times in  $u$  as in  $u'$ . It is easy to see that every word of length eight over an alphabet of three letters contains an Abelian square. Evdokimov was the first to prove that there is an infinite word on a finite alphabet (in fact, on 25 letters) that contains no Abelian squares; a little later Pleasants lowered Evdokimov's 25 to 5.

A.A. Evdokimov, Strongly asymmetric sequences generated by a finite number of symbols (in Russian), *Dokl. Akad. Nauk SSSR* **179** (1968), 1268–1271. (See also *Soviet Math. Dokl.* **9** (1968), 536–539.)

M. Lothaire, *Combinatorics on Words*, A collective work by Dominique Perrin, Jean Berstel, Christian Choffrut, Robert Cori, Dominique Foata, Jean Eric Pin, Guiseppe Pirillo, Christophe Reutenauer, Marcel-P. Schützenberger, Jacques Sakarovitch and Imre Simon, with a foreword by Roger Lyndon, edited and with a preface by Dominique Perrin, *Encyclopedia of Mathematics and its Applications*, **17**, Addison-Wesley, (1983), xix+238 pp. See also M. Lothaire, *Combinatorics on Words*; corrected reprint of the 1983 original, with a new preface by Dominique Perrin, *Cambridge Mathematical Library*, Cambridge University Press, (1997), xviii + 238 pp.

M. Morse, Recurrent geodesics on a surface of negative curvature, *Trans. Amer. Math. Soc.* **22** (1921), 84–100.

M. Morse, A solution of an infinite play in chess, *Bull. Amer. Math. Soc.* **44** (1938), 642.

M. Morse and G.A. Hedlund, Symbolic dynamics, *Amer. J. Math.* **60** (1938), 815–866.

M. Morse and G.A. Hedlund, Unending chess, symbolic dynamics and a problem in semi-groups, *Duke Math. J.* **11**, (1944). 1–7.

P.A. Pleasants, Non-repetitive sequences, *Math. Proc. Cambridge Phil. Soc.* **68** (1970), 267–274.

A. Thue, Über unendliche Zeichenreihen (in German), *Norske Vid. Selsk. Skr. I. Mat. Nat. Kl.*, Christiania, (1906), no. 7, 1–22.

A. Thue, Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen (in German), *Norske Vid. Selsk. Skr. I. Mat. Nat. Kl.*, Christiania, (1912), no. 1, 1–67.

## 96. Addition of Residue Classes: the Cauchy–Davenport Theorem

(i) Let  $A$  and  $B$  be non-empty subsets of  $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ , the set of congruence classes modulo a prime  $p \geq |A| + |B| - 1$ . Then

$$A + B = \{a + b : a \in A, b \in B\}$$

has at least  $|A| + |B| - 1$  elements.

(ii) For a prime  $p$  and integer  $k$ ,  $1 \leq k \leq p - 1$ , let  $G_{p,k}$  be a circulant digraph on  $\mathbb{Z}_p$  with out-degree  $k$ . Then every set of  $p - k + 1$  vertices of  $G_{p,k}$  induces a strongly connected subgraph.

*Proof.* (i) Let us apply induction on  $m$ , the minimum of  $|A|$  and  $|B|$ . The assertion is obvious if  $m = 1$ . Indeed, if, say,  $B$  has one element,  $b$ , then  $|A + B| = |\{a + b : a \in A\}| = |A| = |A| + |B| - 1$ . Suppose then that  $m > 1$ , say,  $1 < m = |B| \leq |A|$ , and the assertion holds for smaller values of  $m$ .

Let us make the ‘reasonable assumption’ that  $B \cap A$  is neither empty nor the entire set  $B$ , i.e., the set  $B$  (which is at most as large as  $A$ ) has an element that belongs to  $A$ , and also an element that does not belong to  $A$ . In this case we are easily done by the induction hypothesis: with  $A' = A \cup B$  and  $B' = A \cap B$  we have

$$\begin{aligned} |A'| + |B'| &= |A| + |B| \leq p + 1, \\ A' + B' &\subset A + B, \end{aligned}$$

and so

$$|A + B| \geq |A' + B'| \geq |A'| + |B'| - 1 = |A| + |B| - 1,$$

proving the induction step.

Hence, to complete the proof, it suffices to justify our ‘reasonable assumption’. To this end, note that  $|A + B|$  does not change if we multiply *both*  $A$  and  $B$  by the same non-zero number  $q \in \mathbb{Z}_p$ , neither does it change if we translate  $A$  by a number  $r$  and  $B$  by a number  $s$ . Putting it more formally: if  $0 \neq q \in \mathbb{Z}$  then with  $A' = qA + r$  and  $B' = qB + s$  we have  $|A'| = |A|$ ,  $|B'| = |B|$  and  $|A' + B'| = |A + B|$ . In view of this, we may assume that  $B$  contains two consecutive elements  $b$  and  $b + 1$ , say. To see that this can be assumed, pick two distinct elements of  $B$ ,  $b_1$  and  $b_2$ , say, and multiply both  $A$  and  $B$  by  $q = (b_2 - b_1)^{-1}$ . Next, since  $A$  is not the entire set  $\mathbb{Z}_p$ , there is an  $a \in A$  such that  $a + 1 \notin A$ . Replacing  $A$  by its translate  $A + (b - a)$ , we find that  $b \in A$  and  $b + 1 \notin A$ . But then  $b \in B \cap A$

and  $b + 1 \in B \setminus A$ , so  $1 \leq |A \cap B| < |B|$ . Hence our ‘reasonable assumption’ is justified, and so our proof is complete.

(ii) This part is an easy consequence of part (i). Indeed, let  $G_{p,k}$  have vertex set  $\mathbb{Z}_p$ , and let  $C \subset \mathbb{Z}_p \setminus \{0\}$  be its set of chords. [Thus  $\overrightarrow{ab}$  is an edge (going from  $a$  to  $b$ ) if and only if  $b - a \in C$ .] Suppose for a contradiction that there is a subgraph  $H$  of  $G_{p,k}$  which is not strongly connected. Then  $V(H)$  has a partition  $A \cup B$  into non-empty subsets such that no edge goes from  $A$  to  $B$ , i.e.,

$$B - A = \{b - a : b \in B, a \in A\}$$

is disjoint from  $C$ . Since  $0 \in B - A$ , this implies that

$$(B + (-A)) \cap (C \cup \{0\}) = \emptyset,$$

and so

$$|B + (-A)| \leq p - (k + 1).$$

However, this is impossible since, by (i),

$$|B + (-A)| \geq |B| + |A| - 1 = p - k.$$

contradicting the previous inequality.  $\square$

**Notes.** The first assertion above is the *Cauchy–Davenport theorem*, one of the fundamental results in additive number theory. The rather surprising name is due to the fact that although Davenport published a proof of it in 1935, twelve years later, in ‘*A historical note*’, he drew attention to the fact that he had been anticipated by Cauchy by hundred and twenty two years. Indeed, in 1813 Cauchy gave essentially the first proof above. The problem arose as the ‘mod  $p$  analogue’ of a conjecture of Khintchine concerning the density of the sum of two sequences of integers; this conjecture was solved later by Mann, and is now known as the  $\alpha + \beta$  theorem of Mann. For various extensions and ramifications of the Cauchy–Davenport result see Chowla, Pollard, and Alon and Dubiner. For example, Pollard proved that if  $A, B \subset \mathbb{Z}_p$  with  $1 \leq |B| \leq |A| \leq p + r - |B|$ , where  $1 \leq r \leq |B|$ , and  $N_r$  is the number of elements of  $\mathbb{Z}_p$  representable in at least  $r$  ways as  $a + b$ ,  $a \in A$ ,  $b \in B$ , then

$$N_1 + N_2 + \cdots + N_r \geq r(|A| + |B| - r).$$

For  $r = 1$  this is just the Cauchy–Davenport inequality.

The second part was noted by Davenport in his 1935 paper, although he formulated it slightly differently, without mentioning graphs. He also pointed out that the two parts are equivalent.

N. Alon and M. Dubiner, Zero-sum sets of prescribed size, in *Combinatorics, Paul Erdős is Eighty* 1, Bolyai Soc. Math. Stud., János Bolyai Math. Soc., Budapest, (1993), pp. 33–50.

- A. Cauchy, Recherches sur les nombres, *Journal de l'Ecole Polytechnique* **9**, (1813), 99–123. Reproduced in *Oeuvres Complètes*, Série 2, Tome 1, Paris, (1905), pp. 39–63.
- I. Chowla, A theorem on the addition of residue classes, *Proc. Indian Acad. Sci.* **2** (1935), 242–243.
- H. Davenport, On the addition of residue classes, *J. London Math. Soc.* **10** (1935), 30–32.
- H. Davenport, A historical note, *J. London Math. Soc.* **22** (1947), 100–101.
- H. Halberstam and K.F. Roth, *Sequences*, Vol. I, Clarendon Press, (1966), xx + 291 pp.
- C. Loewner and H.B. Mann, On the difference between the geometric and the arithmetic mean of  $n$  quantities, *Advances in Math.* **5** (1970), 472–473.
- H.B. Mann, Two addition theorems, *J. Combin. Theory* **3** (1967), 233–235.
- H.B. Mann, *Addition Theorems: the Addition Theorems of Group Theory and Number Theory*, Reprint, with corrections, of the 1965 original, Robert E. Krieger Publishing Co., (1976), xi + 114 pp.
- J.M. Pollard, A generalisation of a theorem of Cauchy and Davenport, *J. London Math. Soc.* (2) **8** (1974), 460–462.
- J.M. Pollard, Addition properties of residue classes, *J. London Math. Soc.* (2) **11** (1975), 147–152.

## 97. Sums Congruent to Zero: the Erdős–Ginzburg–Ziv Theorem

*Every sequence of  $2n - 1$  natural numbers contains  $n$  terms whose sum is divisible by  $n$ . This assertion is false if  $2n - 1$  is replaced by  $2n - 2$ .*

*First Proof.* Note first that a sequence consisting of  $n - 1$  0s and  $n - 1$  1s does not have  $n$  terms whose sum is divisible by  $n$ , so the assertion is certainly false with  $2n - 1$  replaced by  $2n - 2$ .

Let us prove the real assertion in the case when  $n$  is a prime: *if  $p$  is a prime and  $a_1, a_2, \dots, a_{2p-1}$  is a sequence of integers then the sum of some  $p$  terms is divisible by  $p$ .* As we are interested in divisibility by  $p$ , we may assume that  $0 \leq a_1 \leq a_2 \leq \dots \leq a_{2p-1} < p$ . If, for some  $i$ ,  $1 \leq i \leq p - 1$ , we have  $a_{i+p-1} = a_i$  then all  $p$  terms  $a_i, a_{i+1}, \dots, a_{i+p-1}$  are equal, so their sum is a multiple of  $p$ . Therefore we may suppose that  $a_i \neq a_{i+p-1}$  for  $i = 1, \dots, i + p - 1$ .

We claim that in this case considerably more is true than the assertion we have to prove: for every integer  $s$  there is a set of  $p$  terms of our sequence whose sum is congruent to  $s$  modulo  $p$ . To show this, let  $A_i = \{a_i, a_{i+p-1}\}$ ,  $i = 1, \dots, p - 1$  and  $A_p = \{a_{2p-1}\}$  be subsets of  $\mathbb{Z}_p$ . We shall prove that  $A_1 + A_2 + \dots + A_p = \mathbb{Z}_p$ , i.e., every element of  $\mathbb{Z}_p$  is the sum of  $p$  terms, one from each set  $A_i$ . This will follow if we prove that for each  $i$ ,  $1 \leq i < p$ , the set  $B_i = A_1 + A_2 + \dots + A_i$  has at least  $i + 1$  elements.

Suppose that this is false. Then for some  $i$ ,  $1 \leq i < p - 1$ , the sets  $B_i$  and  $B_{i+1}$  have  $i + 1$  elements each. To simplify the notation, set  $B = B_i$  and  $A_{i+1} = \{c, d\}$ . Then the sets  $C = B + c$  and  $D = B + d$  coincide, i.e., the set  $C$  is invariant under the addition of  $e = d - c$ : if  $x$  is in  $C$ , then so are  $x + e$ ,  $x + 2e$ , etc. But then, since  $p$  is prime,  $C$  is the entire set  $\mathbb{Z}_p$ , contradicting our assumption that  $|C| = |B_i| = i + 1 < p$ . This contradiction completes our proof in the case when  $p$  is a prime.

The general case follows easily by induction on  $n$ . Indeed, for  $n = 1$  there is nothing to prove. Suppose that  $n > 1$  and the assertion holds for smaller values of  $n$ . Let  $p$  be a prime factor of  $n$  and set  $m = n/p$ . Successively select disjoint  $p$ -subsets  $I_1, \dots, I_{2m-1}$  of  $I = \{1, \dots, 2n - 1\}$  such that each  $b_j = \sum_{i \in I_j} a_j$ ,  $j = 1, \dots, 2m - 1$ , is a multiple of  $p$ . This can be done since when we have selected  $\ell < 2m - 1$  such sets, we still have a sequence of  $2n - 1 - \ell p \geq 2n - 1 - (2m - 2)p = 2p - 1$  elements to choose from.

Finally, consider the integer sequence  $c_1, \dots, c_{2m-1}$ , where  $c_i = b_i/p$ . Since  $m < n$ , by the induction hypothesis there is an  $m$ -set  $J \subset \{1, \dots, 2m - 1\}$  such that  $\sum_{j \in J} c_j$  is divisible by  $m$ . Consequently, the  $n$ -set  $I = \bigcup_{j \in J} I_j \subset \{1, 2, \dots,$



$2n - 1\}$  is such that  $\sum_{i \in I} a_i = p \sum_{j \in J} \sum_{i \in I_j} a_i / p = p \sum_{j \in J} c_j$  is a multiple of  $p$ ,  $pm = n$ , as claimed.  $\square$

*Second Proof.* Let us give another proof in the main case, i.e., when  $n$  is a prime  $p$ . Thus, let  $a_1, \dots, a_{2p-1}$  be a sequence of  $2p - 1$  elements of  $\mathbb{Z}_p$ . Let us evaluate the sum

$$S = \sum_{I \subset \mathbb{Z}_p, |I|=p} \left( \sum_{i \in I} a_i \right)^{p-1}$$

in two different ways. First, considering  $S$  as a polynomial over  $\mathbb{Z}$  in the variables  $a_1, \dots, a_{2p-1}$ , write  $S$  as the sum of monomials of the form

$$c_{\mathbf{k}} \prod_{i \in I} a_i^{k_i},$$

where  $\mathbf{k} = (k_i)_1^p$  is a sequence of  $p$  non-negative integers  $k_i$  summing to  $p - 1$ . Let  $|\mathbf{k}|$  be the number of non-zero terms of the sequence  $\mathbf{k}$ , so that  $1 \leq |\mathbf{k}| \leq p - 1$ . Clearly, a  $p$ -set  $I$  is such that  $(\sum_{i \in I} a_i)^{p-1}$  contributes to  $c_{\mathbf{k}}$  if  $I$  contains the  $|\mathbf{k}|$  non-zero indices  $i$  with  $k_i > 0$ , and each such  $p$ -set contributes the same natural number. Since there are

$$\binom{2p-1-|\mathbf{k}|}{p-|\mathbf{k}|} \equiv 0 \pmod{p}$$

such sets  $I$ , the coefficient  $c_{\mathbf{k}}$  is a multiple of  $p$  so, *a fortiori*,  $S = 0$  in  $\mathbb{Z}_p$ .

Now, let us evaluate  $S$  in a different way, supposing that the assertion fails, i.e.,  $\sum_{i \in I} a_i \neq 0$  whenever  $I \subset \mathbb{Z}_p$  and  $|I| = p$ . Then, by Fermat's little theorem,  $(\sum_{i \in I} a_i)^{p-1} = 1$  for every  $p$ -subset  $I$  of  $\mathbb{Z}_p$  so, since

$$\binom{2p-1}{p} \equiv 1 \pmod{p},$$

we see that  $S = 1$ . This contradiction completes our second proof.  $\square$

**Notes.** The assertion is the celebrated and extremely influential *Erdős–Ginzburg–Ziv theorem*, proved in 1961. The first proof above is the original; the second, due to N. Zimmermann, was reported by Alon and Dubiner in a beautiful survey of some of the many results related to the Erdős–Ginzburg–Ziv theorem.

The heart of the original proof is the assertion that if  $A_1, A_2, \dots, A_{p-1}$  are subsets of  $\mathbb{Z}_p$ , each with two elements, then  $A_1 + A_2 + \dots + A_{p-1} = \mathbb{Z}_p$ . This is immediate from the Cauchy–Davenport theorem (see Problem 96); in fact, as shown above, this inequality is a consequence of the first non-trivial case of the Cauchy–Davenport theorem that if  $A$  and  $B$  are subsets of  $\mathbb{Z}_p$ , with  $2 \leq |A| \leq p-1$  and  $|B| = 2$  then  $|A + B| \geq |A| + 1$ .

Alon and Dubiner presented five proofs of the and Erdős–Ginzburg–Ziv theorem, including the two given above. A third proof, due to Alon, is based on the following theorem of Chevalley and Warning on polynomials over a finite field.

Let  $p$  be a prime, and for  $j = 1, \dots, n$ , let  $P_j(X_1, \dots, X_m)$  be a polynomial over  $\mathbb{Z}_p$  with degree  $d_j$  such that

$$\sum_{j=1}^n d_j < m.$$

Then the number of common zeros of  $P_1, \dots, P_n$  in  $\mathbb{Z}_p^m$  is divisible by  $p$ . In particular, if the polynomials have a common zero then they have at least two common zeros.

We leave the easy proof of this result to the reader and rather show how Alon deduced from it (the heart of) the Erdős–Ginzburg–Ziv theorem.

*Third Proof.* As before, let  $p$  be a prime and let  $a_1, \dots, a_{2p-1}$  be a sequence of elements of  $\mathbb{Z}_p$ . Define polynomials  $P_1$  and  $P_2$  over  $\mathbb{Z}_p$  as follows:

$$P_1(X_1, \dots, X_{2p-1}) = \sum_{i=1}^{2p-1} X_i^{p-1} \quad \text{and} \quad P_2(X_1, \dots, X_{2p-1}) = \sum_{i=1}^{2p-1} a_i X_i^{p-1}.$$

Clearly,  $\mathbf{0} = (0, \dots, 0) \in \mathbb{Z}_p^{2p-1}$  is a zero of both  $P_1$  and  $P_2$ . Also, the sum of the degrees of  $P_1$  and  $P_2$  is  $2(p-1)$ , which is less than the number of variables,  $2p-1$ . Hence, by the Chevalley–Warning theorem,  $P_1$  and  $P_2$  have another (this time, non-trivial) common zero  $\mathbf{z} = (z_1, \dots, z_{2p-1})$ . Let  $I = \{i : z_i \neq 0\}$ , and note that, by the little Fermat theorem,  $z_i^{p-1} = 1$  if  $i \in I$  and  $z_i^{p-1} = 0$  if  $i \notin I$ . Consequently,  $P_1(\mathbf{z}) = \sum_{i \in I} 1 = 0$  and  $P_2(\mathbf{z}) = \sum_{i \in I} a_i = 0$ , i.e.,  $|I| \neq 0$  is a multiple of  $p$  and  $\sum_{i \in I} a_i = 0$ . Since  $0 < |I| < 2p-1$ , we see that  $|I| = p$  and  $\sum_{i \in I} a_i = 0$ , as claimed.  $\square$

The following beautiful extension of the Erdős–Ginzburg–Ziv theorem was conjectured by Kemnitz in 1981: every sequence of lattice points in  $\mathbb{Z}^2$  consisting of  $4n-3$  terms contains  $n$  terms whose sum has coordinates divisible by  $n$ . Kemnitz himself proved that if the conjecture is true for  $n$  and  $m$  then it is true for  $nm$  as well; in particular, it suffices to prove the conjecture in the case when  $n = p$  for some prime  $p$ . He also checked that his conjecture holds for  $p = 2, 3, 5$  and  $7$ .

Alon and Dubiner made use of the Chevalley–Warning theorem to prove that the conjecture is true with  $4p-3$  replaced by  $6p-5$  and, for  $p$  large, by  $5p-2$ . Rónyai came tantalizingly close to proving Kemnitz’s conjecture when he showed that the assertion is true for  $4p-2$  instead of  $4p-3$ . Finally, in October 2003, Reiher, an undergraduate, gave an elegant and simple proof of the conjecture by making clever use of the Chevalley–Warning theorem. According to Savchev and

Chen, at the same time, C. di Fiore, a high school student, independently found a similar proof.

In a different direction, Bollobás and Leader proved that if  $G$  is an Abelian group of order  $n$ ,  $1 \leq r \leq n-1$ ,  $a_1, \dots, a_{k+r} \in G$  and 0 is not an  $n$ -sum, then the number of  $n$ -sums is at least  $r+1$ . The case  $r = n-1$  clearly implies the Erdős–Ginzburg–Ziv theorem. The original proof of this result was not too complicated, but Yu found an even simpler proof.

There are numerous other developments stemming from the Erdős–Ginzburg–Ziv theorem: for a sample of these, see the references below.

N. Alon and M. Dubiner, Zero-sum sets of prescribed size, in *Combinatorics, Paul Erdős is Eighty*, Vol. 1, Bolyai Soc. Math. Stud., János Bolyai Math. Soc., Budapest, (1993), pp. 33–50.

A. Bialostocki, G. Bialostocki and D. Schaal, A zero-sum theorem, *J. Combin. Theory Ser. A* **101** (2003), 147–152.

A. Bialostocki and P. Dierker, On the Erdős–Ginzburg–Ziv theorem and the Ramsey numbers for stars and matchings, *Discrete Math.* **110** (1992), 1–8.

A. Bialostocki, P. Dierker, D. Grynkiewicz and M. Lotspeich, On some developments of the Erdős–Ginzburg–Ziv theorem, II, *Acta Arith.* **110** (2003), 173–184.

B. Bollobás and I. Leader, The number of  $k$ -sums modulo  $k$ , *J. Number Theory* **78** (1999), 27–35.

P. Erdős, A. Ginzburg and A. Ziv, Theorem in additive number theory, *Bull. Res. Council Israel* **10F** (1961), 41–43.

C. Flores and O. Ordaz, On the Erdős–Ginzburg–Ziv theorem, *Discrete Math.* **152** (1996), 321–324.

L. Gallardo, and G. Grekos, On Brakemeier’s variant of the Erdős–Ginzburg–Ziv problem, in *Number Theory*, (Liptovský Ján, 1999), Tatra Mt. Math. Publ. **20** (2000), 91–98.

L. Gallardo, G. Grekos, L. Habsieger, F. Hennecart, B. Landreau and A. Plagne, Restricted addition in  $\mathbb{Z}/n\mathbb{Z}$  and an application to the Erdős–Ginzburg–Ziv problem, *J. London Math. Soc.* (2) **65** (2002), 513–523.

L. Gallardo, G. Grekos and J. Pihko, On a variant of the Erdős–Ginzburg–Ziv problem, *Acta Arith.* **89** (1999), 331–336.

W.D. Gao, An improvement of the Erdős–Ginzburg–Ziv theorem (in Chinese), *Acta Math. Sinica* **39** (1996), 514–523.

W.D. Gao, On the Erdős–Ginzburg–Ziv theorem – a survey, in *Paul Erdős and his Mathematics* (Budapest, 1999), János Bolyai Math. Soc., Budapest, (1999), pp. 76–78.

Y.O. Hamidoune, On weighted sums in abelian groups, *Discrete Math.* **162** (1996), 127–132.

Y.O. Hamidoune, O. Ordaz, and A. Ortuño, On a combinatorial theorem of Erdős, Ginzburg and Ziv, *Combin. Prob. and Comput.* **7** (1998), 403–412.

A. Kemnitz, On a lattice point problem *Ars Combin.* **16** (1983), 151–160.

C. Reiher, Kemnitz’ conjecture concerning lattice-points in the plane, to appear.

L. Rónyai, On a conjecture of Kemnitz, *Combinatorica* **20** (2000), 569–573.

S. Savchev and F. Chen, Kemnitz’ conjecture revisited, *Discrete Math.* **297** (2005), 196–201.

Z.-W. Sun, Unification of zero-sum problems, subset sums and covers of  $\mathbb{Z}$ , *Electron. Res. Announc. Amer. Math. Soc.* **9** (2003), 51–60 (electronic).

C. Wang, Note on a variant of the Erdős–Ginzburg–Ziv problem, *Acta Arith.* **108** (2003), 53–59.

H.B. Yu, A simple proof of a theorem of Bollobás and Leader, *Proc. Amer. Math. Soc.* **131** (2003), 2639–2640.



## 98. Subwords of Distinct Words

Let  $u$  and  $u'$  be distinct words of length  $\ell \geq m$ . If  $\ell \leq 2m - 1$  then some word of length  $m$  is a subword of one but not of the other, but this need not hold if  $\ell \geq 2m$ .

*Proof.* That the condition  $\ell \leq 2m - 1$  is necessary is trivial. Indeed, let  $u = abab \dots ab$  and  $w = baba \dots ba$  be words of length  $\ell = 2m$ . Then each of  $u$  and  $w$  contains all  $2^m$  words of length  $m$  on  $a$  and  $b$ . As another example, take  $u = a^{m-1}ba^m$  and  $w = a^mba^{m-1}$ .

Turning to the main assertion, let  $u$  and  $w$  be distinct words of length  $\ell = 2m - 1$  on  $a$  and  $b$ . Suppose for a contradiction that they have precisely the same subwords of length  $m$ . Then, in particular,  $a$  occurs in  $u$  the same number of times as in  $w$ , and so does  $b$ . (Indeed, otherwise we may assume that  $a$  occurs  $r < m$  times in  $u$  and more than  $r$  times in  $w$ . But then  $w$  has a subword of length  $m$  containing  $a$  at least  $r + 1$  times, while  $u$  does not.)

Suppose  $a$  occurs  $\alpha$  times in  $u$  and  $b$  occurs  $\beta = \ell - \alpha$  times. Write  $u$  and  $w$  in the following canonical form:

$$u = a^{i_0}b a^{i_1}b \dots a^{i_{\beta-1}}b a^{i_{\beta}} \quad \text{and} \quad w = a^{j_0}b a^{j_1}b \dots a^{j_{\beta-1}}b a^{j_{\beta}},$$

where  $\sum_{s=0}^{\beta} i_s = \sum_{s=0}^{\beta} j_s = \alpha$ .

Since  $u \neq w$ , we have  $i_s \neq j_s$  for some  $s$ : let  $k$  be the smallest such index. Thus  $i_s = j_s$  for  $s < k$ , and  $i_k \neq j_k$ ; we may and shall assume that  $i_k < j_k$ , so that  $\sum_{s=k+1}^{\beta} i_s > \sum_{s=k+1}^{\beta} j_s$ . Let us define two words that we claim distinguish  $u$  and  $w$ :

$$x = a^{i_0+i_1+\dots+i_k+1}b^{\beta-k} \quad \text{and} \quad y = b^{k+1}a^{j_{k+1}+j_{k+2}+\dots+j_{\beta}}.$$

Indeed,  $x \not\leq u$  but  $x \leq w$ , and  $y \leq u$  but  $y \not\leq w$ , where  $z' \leq z$  means that  $z'$  is a subword of  $z$ . Consequently, each of  $x$  and  $y$  has length at least  $m + 1$ , i.e.,

$$i_0 + \dots + i_k + 1 + \beta - k \geq m + 1 \quad \text{and} \quad k + 1 + j_{k+1} + \dots + j_{\beta} + 1 \geq m + 1.$$

Summing these equations, we find that

$$\begin{aligned} \ell &= (i_0 + \dots + i_k + \beta - k) + (k + i_{k+1} + \dots + i_{\beta}) \\ &\geq (i_0 + \dots + i_k + \beta - k) + (k + 1 + j_{k+1} + \dots + j_{\beta}) \geq 2m. \end{aligned}$$

This contradiction completes our proof. □

**Notes.** Although the result above is due to M.-P. Schützenberger and I. Simon, it first appeared in the *Subwords* chapter of *Lothaire* (see the reference in Problem 95).

## 99. Prime Factors of Sums

Let  $S$  consist of  $2^k + 1 \geq 3$  natural numbers. Then the two-term sums  $a + b$ ,  $a, b \in S$ , cannot all be composed of  $k$  given prime numbers.

*Proof.* For a prime  $p$  and integer  $a$ , write  $v_p(a)$  for the exponent of  $p$  in the prime factorization of  $a$ . Thus  $v_2(72) = 3$ ,  $v_3(72) = 2$  and  $v_5(72) = 0$ . A set  $S \subset \mathbb{N}$  is said to be  $p$ -simple if  $v_p(a + b) = \min\{v_p(a), v_p(b)\}$  for all  $a, b \in S$ ,  $a \neq b$ . First, we prove the following claim.

**Claim.** Let  $p \geq 3$  be a prime, and  $A = \{a_1, a_2, \dots, a_n\} \subset \mathbb{N}$ . Then  $A$  has a  $p$ -simple subset  $S$  with at least  $n/2$  elements.

*Proof of Claim.* Set  $a_i = p^{\alpha_i} b_i$ , with  $\alpha_i = v_p(a_i)$ , so that  $p$  does not divide  $b_i$ , say,  $b_i = q_i p + r_i$ , where  $1 \leq r_i \leq p - 1$ . With  $I_0 = \{i : r_i < p/2\}$  and  $I_1 = \{i : r_i > p/2\}$  we have  $I_0 \cup I_1 = [n]$ . If  $|I_0| \geq |I_1|$ , choose  $I_0$  for  $I$ , otherwise set  $I = I_1$ . Then  $S = \{a_i : i \in I\}$  is  $p$ -simple.  $\square$

Armed with this Claim, the assertion is easily proved. Suppose for a contradiction that all  $n(n - 1)/2$  two-term sums of the elements of a set  $S \subset \mathbb{N}$ ,  $|S| = 2^k + 1 \geq 3$ , are composed of the primes  $p_1 = 2, p_2, \dots, p_k$ . (Since  $S$  has at least three elements, one of the primes has to be 2.) By the Claim,  $S_k = S$  contains a  $p_k$ -simple set  $S_{k-1}$  with  $2^{k-1} + 1$  numbers, then  $S_{k-1}$  contains a  $p_{k-1}$ -simple set  $S_{k-2}$  with  $2^{k-2} + 1$  numbers, and so on. Therefore  $S$  contains a set  $S_1 = \{a_1, a_2, a_3\}$  which is  $p_i$  simple for every  $i$ ,  $2 \leq i \leq k$ .

To simplify the notation, for a sequence of integers  $\mathbf{s} = (s_2, \dots, s_k)$ , we write  $p^{\mathbf{s}} = p_2^{s_2} \dots p_k^{s_k}$ . In this notation,

$$\begin{aligned} a_1 + a_2 &= 2^h p^{\mathbf{h}}, \\ a_1 + a_3 &= 2^i p^{\mathbf{i}}, \\ a_2 + a_3 &= 2^j p^{\mathbf{j}}, \end{aligned}$$

for some numbers  $h, i, j$  and sequences  $\mathbf{h}, \mathbf{i}, \mathbf{j}$ . Then  $a_1$  and  $a_2$  are divisible by  $p^{\mathbf{h}}$ ; therefore  $v_2(a_1) = v_2(a_2)$ . Indeed, otherwise, let  $g$  be the minimum of  $v_2(a_1)$  and  $v_2(a_2)$ . Then

$$a_1/2^g p^{\mathbf{h}} + a_2/2^g p^{\mathbf{h}},$$

being the sum of an odd number and an even number, is odd, so has a  $p_i$ -factor, contradicting the fact that it is  $2^{h-g}$ . Similarly,  $v_2(a_1) = v_2(a_3)$ , so  $v_2(a_1) = v_2(a_2) = v_2(a_3) = g$ , say.

We have discovered enough about the numbers  $a_1$ ,  $a_2$  and  $a_3$  to enable us to arrive at a contradiction. Indeed, set  $b_i = a_i/2^g$ , so that  $b_1, b_2, b_3$  are odd numbers and

$$b_1 + b_2 = 2^{h'} p^h,$$

$$b_1 + b_3 = 2^{i'} p^i,$$

$$b_2 + b_3 = 2^{j'} p^j$$

for  $h' = h - g$ ,  $i' = i - g$ ,  $j' = j - g \geq 0$ . Note that  $h' \geq 2$  since  $b_1/p^h$  and  $b_2/p^h$  are different odd numbers summing to  $2^{h'}$ . Similarly,  $i' \geq 2$  and  $j' \geq 2$ . Therefore,  $b_1, b_2$  and  $b_3$  are odd integers such that the sum of any two is divisible by 4. This is impossible, completing our proof.  $\square$

**Notes.** This is an early result of Paul Erdős and Paul Turán (On a problem in the elementary theory of numbers, *Amer. Math. Monthly* **41** (1934), 608–611), solving a problem of G. Grünwald and D. Lázár.

## 100. Catalan Numbers

The  $n$ th Catalan number  $C_n = \frac{1}{n+1} \binom{2n}{n}$  enumerates the following sets of combinatorial structures.

- (i) Legal bracketings that use  $n$  pairs of brackets.
- (ii) Sets consisting of  $2n$  points on a circle joined with  $n$  non-intersecting chords, with each point on one chord.
- (iii) Integer sequences  $(a_i)_{i=1}^{n-1}$  such that  $1 \leq a_1 \leq a_2 \leq \dots \leq a_{n-1}$  and  $a_i \leq i$  for every  $i$ .
- (iv) Pairs of sequences  $(a_i)_{i=1}^k, (b_j)_{j=1}^k$  of natural numbers are there such that  $\sum_{i=1}^k a_i = \sum_{j=1}^k b_j = n$  and  $\sum_{j=1}^h b_j \leq \sum_{i=1}^h a_i$  for every  $h, h = 1, \dots, k$ , with  $k = 1, \dots, n$ .
- (v) Binary plane trees with  $n + 1$  end-vertices.
- (vi) Unordered pairs of monotone increasing lattice paths in  $\mathbb{Z}^2$ , each with  $n + 1$  steps, starting at the origin and ending at the same point, and sharing only the beginning and the end.

*Proof.* Assertions (i)–(iv) are rather trivial, since in each case we can identify the set in question with the set of increasing lattice paths that do not rise above the line  $y = x$ . For example, in (i) an opening bracket ‘(’ is identified with a step to the right and a closing bracket ‘)’ with a step upwards. In (ii), writing  $A_1, A_2, \dots, A_{2n}$  for the points on the circle in a cyclic order, a distribution of chords is mapped into a bracketing by coding a chord  $A_i A_j$  with  $i < j$  with an opening bracket in position  $i$  and a closing bracket in position  $j$ . To see (iii), map an increasing lattice path from  $(0, 0)$  to  $(n, n)$  into the sequence  $(a_i)_{i=1}^{n-1}$ , where  $a_i$  is the number of steps upwards before the  $(i + 1)$ st step to the right. In (iv), a pair  $(a_i)_{i=1}^k, (b_j)_{j=1}^k$  gives the following path starting at  $(0, 0)$ : take  $a_1$  steps to the right, then  $b_1$  steps upwards, then  $a_2$  steps to the right, and so on.

To prove (v), identify the set of binary plane trees with triangulations of a convex  $(n + 2)$ -gon, and invoke the result in Problem 62. This identification is easily seen from Figure 77. An edge of the polygon is fixed and is made to correspond to the root of the tree.

Finally, (vi) needs an idea. Consider the planar domain  $P_n$  bounded by the two paths: this is a so-called *parallelogram polyomino* with perimeter length  $n + 1$ . Thus  $P_n$  is the union of lattice squares whose boundary is a simple curve made up of two increasing lattice paths, each of length  $n + 1$ . Write  $P_n$  as the union of various columns, with  $c_i$  squares in the  $i$ th column, say. The first column, of height  $c_1$ , has its bottom left corner at the origin, the second column, of height  $c_2$ ,



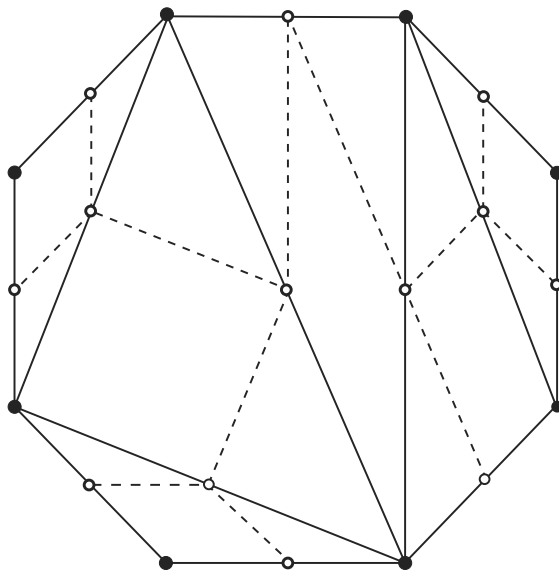


Fig. 77. A triangulation and the binary plane tree it defines. The solid lines and black dots show the triangulation, and the broken lines and open circles show the binary plane tree.

starts no lower than the first but before the first ends, and so on. Write  $d_i$  for the number of rows in which there is a square in column  $i$  and column  $i + 1$ . Thus  $d_i \leq \min(c_i, c_{i+1})$ . Also,  $P_n$  has half perimeter length

$$c_1 + (c_2 - d_1) + (c_3 - d_2) + \cdots + (c_k - d_{k-1}) + k = \sum_{i=1}^k c_i - \sum_{i=1}^{k-1} d_i + k = n + 1.$$

To code  $P_n$ , set  $a_1 = c_1$ ,  $a_i = c_i - d_{i-1} + 1$  for  $i = 2, \dots, k$ , and  $b_i = c_i - d_i + 1$ ,  $i = 1, \dots, k - 1$ ,  $b_k = a_k$ . Then  $\sum_{i=1}^k a_i = \sum_{i=1}^k b_i = n$  and  $\sum_{i=1}^h b_i \leq \sum_{i=1}^h a_i$  for every  $h$ . Conversely, all such pairs  $(a_i)_1^k, (b_i)_1^k$  give a parallelogram polyomino  $P_n$ . Hence, by part (iv), the number in question is indeed  $C_n$ .  $\square$

**Notes.** There is a very rich literature concerning combinatorial structures counted by Catalan numbers: in Problems 61, 62, 100 and 101 we give only a hint of what Stanley calls *Catalania* – Catalan mania; Stanley himself gives sixty-six examples.

R.P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge Studies in Advanced Mathematics **62**, Cambridge University Press, (1999), xii + 581 pp.

## 101. Permutations without Long Decreasing Subsequences

The number of permutations  $a_1 a_2 \dots a_n$  of  $[n]$  that do not contain a decreasing subsequence of length three is the  $n$ th Catalan number  $C_n$ .

*Proof.* Let us code a permutation  $\pi = a_1 a_2 \dots a_n$  of  $[n] = \{1, \dots, n\}$  with the graph  $G(\pi)$  on  $[n]$  in which  $ij$ ,  $i < j$ , is an edge if and only if  $a_i > a_j$ . Clearly,  $G(\pi)$  determines  $\pi$ . In fact,  $\pi$  is also determined by the sequence  $(d_i)$ , where  $d_i$  is the number of neighbours of  $i$  that come after  $i$ . Indeed, given such a ‘forward degree sequence’  $(d_i)_1^n$ , it is immediate that  $a_1 = d_1 + 1$ , and so  $\pi$  is determined by induction on  $n$ . It is reassuring to note that we have  $n + 1 - i$  choices for  $d_i$ , namely  $0, 1, \dots, n - 1$ , so there are  $n!$  forward degree sequences, exactly as many as permutations.

Turning to our assertion, call a permutation *good* if it does not contain a decreasing subsequence of length three. Clearly, a permutation  $\pi$  is good if and only if  $G(\pi)$  does not contain vertices  $i < j < h$  with  $ij$  and  $jh$  edges; equivalently,  $\pi$  is good if and only if no vertex is the end-vertex of an edge and the initial vertex of another. Even more importantly for us, if for a permutation  $\pi$  the vertices of positive forward degrees are  $i_1 < i_2 < \dots < i_\ell$ , then  $\pi$  is good if and only if  $i_{j+1} + d_{j+1} > i_j + d_j$  for every  $j$ ,  $1 \leq j < \ell$ .

Indeed, if  $\pi$  is good and  $h_1 < h_2 < \dots < h_m$ ,  $m = n - \ell$ , are the vertices of  $G(\pi)$  with zero forward degrees, then  $a_{i_1} < a_{i_2} < \dots < a_{i_\ell}$  and  $a_{h_1} < a_{h_2} < \dots < a_{h_m}$  so each  $i_j$  is joined to the first  $d_{i_j}$  vertices  $h_k$  that follow it. Also, if  $i_j$  is joined to  $h_k$  then  $h_k < i_j < i_{j+1}$ , so  $i_{j+1}$  is also joined to  $h_k$ . Hence,  $i_j$  is joined to at most  $i_{j+1} - i_j - 1$  vertices that are not neighbours of  $i_{j+1}$  (the vertices  $i_j + 1, i_j + 2, \dots, i_{j+1} - 1$ , and so  $d_{i_j} - d_{i_{j+1}} \leq i_{j+1} - i_j - 1$ , as claimed. The converse is also easily checked.

Given a forward degree sequence  $(d_i)_1^n$  of the graph of a good permutation with non-zero forward degrees at  $i_1 < i_2 < \dots < i_\ell$ , code it with the increasing lattice path  $L(\pi)$  from  $(0, 0)$  to  $(n, n)$  that has its ‘right turns’ at  $(i_1 + d_1 - 1, i_1)$ ,  $(i_2 + d_2 - 1, i_2)$ ,  $\dots$ ,  $(i_\ell + d_\ell - 1, i_\ell)$ , with the last ‘turn’ in the sense that from  $(i_\ell + d_\ell - 1, i_\ell)$  we get to  $(n, n)$  as quickly as possible, i.e., using at most one horizontal line and at most one vertical line, as in Figure 78. Conversely, every monotone increasing lattice path from  $(0, 0)$  to  $(n, n)$  that does not rise above the line  $y = x$  comes in this way from a good permutation. Recalling Problem 61, this one-to-one correspondence shows that the number of good permutations of  $[n]$  is indeed the Catalan number  $C_n$ .  $\square$

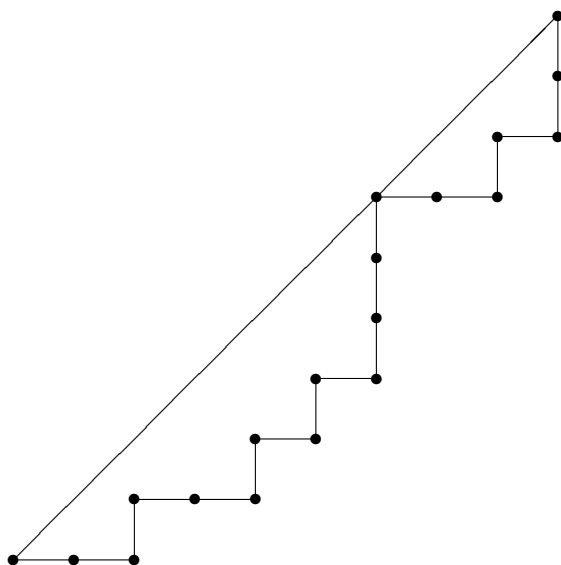


Fig. 78. The graph of  $\pi = 356127948$  has edges 31, 32, 51, 52, 54, 61, 62, 64, 74, 94, 98, so  $i_1 = 1, d_1 = 2, i_2 = 2, d_2 = 3, i_3 = 3, d_3 = 3, i_4 = 6, d_4 = 1, i_5 = 7, d_5 = 2$ , and so the lattice path  $L(\pi)$  is as above.

**Notes.** This result was stated by Hammersley, but Rogers seems to have been the first to publish a proof. The proof above was given by Billey, Jockusch and Stanley; for more references and information see Stanley's book.

S.C. Billey, W. Jockusch and R. Stanley, Some combinatorial properties of Schubert polynomials, *J. Algebraic Combinatorics* **2** (1993), 345–374.

J.M. Hammersley, A few seedlings of research, in *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, (Univ. California, Berkeley, Calif., 1970/1971), Vol. I: *Theory of Statistics*, University of California Press, (1972), pp. 345–394.

D.G. Rogers, Ascending sequences in permutations, *Discrete Math.* **22** (1978), 35–40.

R.P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge Studies in Advanced Mathematics **62**, Cambridge University Press, (1999), xii + 581 pp.

## 102. Random Intervals: a Theorem of Justicz, Scheinerman and Winkler

*Pick  $2n \geq 4$  points uniformly at random in the interval  $[0, 1]$ , say,  $X_1, X_2, \dots, X_{2n}$  and, for  $1 \leq i \leq n$ , let  $J_i$  be the closed interval with end-points  $X_{2i-1}$  and  $X_{2i}$ . Then the probability that there is an interval  $J_i$  that meets all other intervals is  $2/3$ , independently of  $n$ .*

*Proof.* This is surely a very surprising result, unless the reader has taken the trouble to check the answers for  $n = 2$  and  $3$ , say, in which case she must have guessed the answer.

To prove that the probability is indeed  $2/3$ , we construct our intervals as in the *Hint*. By assumption, all cyclic orderings of the  $2n$  objects are equally likely. Given any such cyclic ordering, proceed as follows. Read clockwise from  $X$  until you come across some symbol (say, symbol  $a$ ) for the second time. Turn back anticlockwise and read until you come across some different symbol (say,  $b$ ) twice. Necessarily you have gone beyond  $X$ . Now look at the string of symbols between the  $b$  and the  $a$  at the extremes of this trajectory, running clockwise from the  $b$  to the  $a$ . In this string, look at the order in which  $X$ , the other  $a$ , and the other  $b$ , occur.

All that is known is that the  $a$  comes after the  $X$ : the  $b$  could be:

- (i) before the  $X$ ,
- (ii) between the  $X$  and the  $a$ , or
- (iii) after the  $a$ .

**Claim.** *In cases (ii) and (iii), the interval  $b$  intersects all others, whereas in case (i) there is no interval that intersects all others.*

Of course the point is that the three cases above all occur equally often, as can be seen either by setting up an explicit ‘trijection’ or by partitioning the sample space according to  $a$ ,  $b$ , and the set of symbols encountered.

The Claim is easily proved. Indeed, if the  $b$  is after the  $X$ , and the  $b$ -interval does not intersect some  $c$ -interval, then the relevant symbols occur either as  $bccXb$  or  $bXccb$ , contradicting the choice of  $b$  in the first case and  $a$  in the second. If the  $b$  comes before the  $X$ , and the  $c$ -interval intersects both the  $a$ - and  $b$ -intervals, then both the  $c$ -symbols must lie in the segment, contradicting the choice of  $b$ . This completes our proof: the probability that there is an interval that intersects all others is indeed  $2/3$ . □

**Notes.** The result in this question was proved by Justicz, Scheinerman and Winkler; I heard the beautiful proof above from Graham Brightwell, who has never published it.

J. Justicz, E.R. Scheinerman and P.M. Winkler, Random intervals, *Amer. Math. Monthly* **97** (1990), 881–889.



### 103. Sums of Convex Bodies: the Brunn–Minkowski Inequality

Let  $A$  and  $B$  be bounded convex sets in  $\mathbb{R}^n$  of volume  $a^n$  and  $b^n$ .

(i) Then

$$\text{vol}(A + B) \geq (a + b)^n. \quad (35)$$

(ii) Furthermore, let  $H_t$  be the hyperplane at height  $t$ :

$$H_t = \{(x_1, \dots, x_{n-1}, t) : x_i \in \mathbb{R} \text{ for } i = 1, \dots, n-1\},$$

and write  $v(t)^{n-1}$  for the  $(n-1)$ -dimensional volume of  $A \cap H_t$ . Then the function  $v(t)$  is concave on the interval  $(h_1, h_2)$ , where  $h_1 = \inf\{a_n : (a_1, \dots, a_n) \in A\}$  and  $h_2 = \sup\{a_n : (a_1, \dots, a_n) \in A\}$ .

*Proof.* (i) It suffices to prove (35) when  $A$  is the disjoint union of  $M$  boxes and  $B$  is the disjoint union of  $N$  boxes. Here, as elsewhere, a *box* is a set of the form  $\prod_{i=1}^n [c_i, c'_i] = \{\mathbf{x} = (x_i)_1^n : c_i \leq x_i \leq c'_i, i = 1, \dots, n\}$ . We shall prove this by induction on  $M + N$ . As it happens, the induction step will be very easy to prove, and we shall have to put in a little work only to prove the initial case,  $M = N = 1$ .

When  $M = N = 1$ , we may assume that  $A = [0, 1]^n$  and  $B = \prod_{i=1}^n [0, b_i]$  with  $\text{vol}(A) = 1$  and  $\text{vol}(B) = \prod_{i=1}^n b_i = b^n$ . By the AM-GM inequality, the arithmetic mean of all  $\binom{n}{r}$  products  $b_{i_1} b_{i_2} \cdots b_{i_r}$  is at least the geometric mean  $b^r$  of these products, so

$$\text{vol}(A + B) = \prod_{i=1}^n (1 + b_i) \geq \sum_{r=0}^n \binom{n}{r} b^r = (1 + b)^n,$$

as claimed.

Turning to the induction step, suppose  $M \geq 2$ ,  $N \geq 1$ , and the inequality holds for  $M-1$  and  $N$ . Let  $A = \bigcup_{i=1}^M A_i$  and  $B = \bigcup_{j=1}^N B_j$ , with  $A_i$  and  $B_j$  boxes. One of the faces of the box  $A_M$  determines a hyperplane  $H$  that separates the interior of  $A_M$  from that of  $A_1$ , say. This hyperplane  $H$  cuts  $A$  into two parts,  $A^+$  and  $A^-$ , with  $\text{vol}(A^+) = \alpha a^n$  and  $\text{vol}(A^-) = (1 - \alpha)a^n$  for some  $\alpha$ ,  $0 < \alpha < 1$ . Note that, ignoring sets of measure 0,  $A^+$  is the union of at most  $M-1$  boxes, and so is  $A^-$ .

Let  $H'$  be a hyperplane parallel to  $H$  that cuts  $B$  into sets  $B^+$  and  $B^-$  with  $\text{vol}(B^+) = \alpha b^n$  and  $\text{vol}(B^-) = (1 - \alpha)b^n$ . Clearly, each of  $B^+$  and  $B^-$  is the

union of at most  $N$  boxes. Then  $\text{vol}((A^+ + B^+) \cap (A^- + B^-)) = 0$ , so by the induction hypothesis,

$$\begin{aligned} \text{vol}(A + B) &\geq \text{vol}(A^+ + B^+) + \text{vol}(A^- + B^-) \\ &\geq (\alpha^{1/n}a + \alpha^{1/n}b)^n + ((1 - \alpha)^{1/n}a + (1 - \alpha)^{1/n}b)^n \\ &= \alpha(a + b)^n + (1 - \alpha)(a + b)^n = (a + b)^n, \end{aligned}$$

completing the proof of part (i).

(ii) Let  $S_t$  be the section  $A \cap H_t$  of  $A$  at height  $t$  as a subset of  $\mathbb{R}^{n-1}$ :

$$S_t = \{(a_1, \dots, a_{n-1}) : (a_1, \dots, a_{n-1}, t) \in A\} \subset \mathbb{R}^{n-1}.$$

Thus  $v(t)^{n-1}$  is the  $(n - 1)$ -dimensional volume of  $S_t$ .

Let  $h_1 < t_1 < t_2 < h_2$  and  $t = \lambda t_1 + (1 - \lambda)t_2$ , where  $0 < \lambda < 1$ . By the convexity of  $A$ ,

$$\lambda S_{t_1} + (1 - \lambda)S_{t_2} \subset S_t.$$

Hence, by part (i),  $\lambda v(t_1) + (1 - \lambda)v(t_2) \leq v(t)$ , showing that  $v$  is indeed concave on  $(h_1, h_2)$ .  $\square$

**Notes.** Part (i), the classical *Brunn–Minkowski inequality*, holds whenever  $A$  and  $B$  are measurable sets. Also, equality holds in  $\text{vol}(A + B) \geq (\text{vol}(A)^{1/n} + \text{vol}(B)^{1/n})^n$  if and only if  $A$  and  $B$  are homothetic up to a set of measure 0. Part (ii), an immediate consequence of it, is another basic property of convex sets.

## 104. Cross-Intersecting Families: Bollobás's Lemma

If  $\{(A_i, B_i) : 1 \leq i \leq m\}$  is a cross-intersecting family (of pairs of finite sets) then

$$\sum_{i=1}^m \binom{|A_i| + |B_i|}{|A_i|}^{-1} \leq 1. \quad (36)$$

*First Proof.* We may assume that our finite sets are subsets of  $[n]$ . We shall prove the result by induction on  $n$ . For  $n = 0$  the assertion is trivial as there is only one pair of sets,  $(\emptyset, \emptyset)$ . (Note that for  $n = 1$  there are three possibilities.) Suppose then that  $n \geq 1$  and the inequality is true for smaller values of  $n$ . Let  $\mathcal{S} = \{(A_i, B_i) : 1 \leq i \leq m\}$  be a cross-intersecting family. To avoid the clutter of moduli, we shall write  $a_i$  and  $b_i$  for  $|A_i|$  and  $|B_i|$ . For  $k \in [n]$ , set

$$I_k = \{i : k \notin A_i\} \text{ and } \mathcal{S}_k = \{(A_i, B_i \setminus \{k\}) : i \in I_k\}.$$

Since  $\mathcal{S}_k$  is cross-intersecting, by the induction hypothesis we have

$$\sum_{i \in I_k} \binom{a_i + |B_i \setminus \{k\}|}{a_i}^{-1} \leq 1,$$

so

$$\sum_{k=1}^n \sum_{i \in I_k} \binom{a_i + |B_i \setminus \{k\}|}{a_i}^{-1} = \sum_{i=1}^m \sum_{k \notin A_i} \binom{a_i + |B_i \setminus \{k\}|}{a_i}^{-1} \leq n.$$

Consequently,

$$\begin{aligned} & \sum_{i=1}^m \left\{ \sum_{k \in B_i} \binom{a_i + b_i - 1}{a_i}^{-1} + \sum_{k \notin A_i \cup B_i} \binom{a_i + b_i}{a_i}^{-1} \right\} \\ &= \sum_{i=1}^m \left\{ b_i \binom{a_i + b_i - 1}{a_i}^{-1} + (n - a_i - b_i) \binom{a_i + b_i}{a_i}^{-1} \right\} \leq n. \end{aligned} \quad (37)$$

All that remains is to simplify the summand above to make it obvious that (37) is equivalent to (36). To this end, note that

$$\binom{a_i + b_i}{a_i} = \binom{a_i + b_i}{b_i} = \frac{a_i + b_i}{b_i} \binom{a_i + b_i - 1}{b_i - 1} = \frac{a_i + b_i}{b_i} \binom{a_i + b_i - 1}{a_i}$$

and so

$$b_i \binom{a_i + b_i - 1}{a_i}^{-1} = (a_i + b_i) \binom{a_i + b_i}{a_i}^{-1}$$



Appealing to this identity, (37) becomes

$$\sum_{i=1}^m n \binom{a_i + b_i}{a_i}^{-1} \leq n,$$

showing (36). □

*Second Proof.* As in the first proof, let  $\mathcal{S} = \{(A_i, B_i) : 1 \leq i \leq m\}$  be a cross-intersecting family of subsets of  $[n]$  with  $|A_i| = a_i$  and  $|B_i| = b_i$ . Let  $S_n$  be the symmetric group of all  $n!$  permutations of  $[n]$ , and turn  $S_n$  into a probability space by taking its elements equally probable. For  $1 \leq i \leq m$ , let  $E_i$  be the event that the elements of  $A_i$  precede the elements of  $B_i$ :

$$E_i = \{\pi \in S_n : \pi(a) < \pi(b) \text{ if } a \in A_i \text{ and } b \in B_i\}.$$

Then

$$\mathbb{P}(E_i) = \binom{a_i + b_i}{a_i}^{-1},$$

since  $\pi \in E_i$  if and only if the restriction of  $\pi$  to  $A_i \cup B_i$  maps  $A_i$  into the first  $a_i$  elements of  $\pi(A_i \cup B_i)$ , which is a set with  $a_i + b_i$  elements.

We claim that the events  $E_1, \dots, E_m$  are pairwise-disjoint. Indeed, if  $\pi \in E_i$ , i.e., the set  $\pi(A_i)$  precedes  $\pi(B_i)$  then, for  $j \neq i$ , the set  $\pi(A_j)$  has an element in  $\pi(B_i)$ , and the set  $\pi(B_j)$  has an element in  $\pi(A_j)$ . Consequently,  $\pi(A_j)$  does not precede  $\pi(B_j)$ , so  $\pi \notin E_j$ .

Finally, since the events  $E_1, \dots, E_m$  are pairwise-disjoint,

$$\sum_{i=1}^m \binom{a_i + b_i}{a_i}^{-1} = \sum_{i=1}^m \mathbb{P}(E_i) = \mathbb{P}\left(\bigcup_{i=1}^m E_i\right) \leq 1,$$

completing our proof. □

**Notes.** Bollobás proved this Lemma in 1965; since then various proofs of it and its special cases have been given, notably by Jaeger and Payan in 1971, Katona in 1974 and Tarján in 1975.

As mentioned earlier, a special case of the Lemma is the MYBL inequality presented in Problem 72. Indeed, let  $\mathcal{A}$  be a Sperner family of subsets of  $[n]$ . Setting

$B_A = [n] \setminus A$ , we find that  $(A, B_A)_{A \in \mathcal{A}}$  is a cross-intersecting family so, by the Lemma,

$$\sum_{A \in \mathcal{A}} \binom{|A| + |B_A|}{|A|}^{-1} = \sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1} \leq 1.$$

Let us note the following reformulation that is frequently useful in applications.

Let  $C_i, D_i$ ,  $i = 1, \dots, m$ , be pairs of disjoint subsets of  $[n]$  such that  $C_i \not\subset C_j \cup D_j$  for  $i \neq j$ . Then

$$\sum_{i=1}^m \binom{n - |D_i|}{|C_i|}^{-1} \leq 1. \quad (38)$$

To see this, set  $A_i = C_i$  and  $B_i = [n] \setminus C_i \cup D_i$ . Then  $A_i \cap B_i = \emptyset$ , and  $A_i \cap B_j \neq \emptyset$  if and only if  $C_i \not\subset C_j \cup D_j$ , so (36) holds if and only if so does (38).

In fact, the original formulation of the Lemma was in this asymmetric form; the symmetric formulation of inequality (36) for cross-intersecting pairs was first given by Tuza, who has also given numerous applications of the Lemma.

There are numerous variants and extensions of the Lemma. For example, Bollobás proved a multipartite extension of the Lemma. More importantly, Lovász generalized the uniform version of the result to flats of rank  $r$  of a matroid representable over a field. In particular, he proved that if  $A_1, \dots, A_m$  are  $a$ -dimensional and  $B_1, \dots, B_m$  are  $b$ -dimensional subspaces of a vector space such that  $\dim(A_i \cap B_j) = 0$  if and only if  $i = j$ , then  $m \leq \binom{a+b}{a}$ . This result was further extended by Füredi, who also proved the beautiful result that if  $A_1, \dots, A_m$  are  $a$ -element sets and  $B_1, \dots, B_m$  are  $b$ -element sets such that, for some integer  $t$ ,  $|A_i \cap B_j| \leq t$  if and only if  $i = j$ , then  $m \leq \binom{a+b-2t}{a-t}$ . Further substantial extensions of the Lemma have been proved by Zhu, Talbot, and others.

B. Bollobás, On generalized graphs, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 447–452.

B. Bollobás, The number of unrelated partitions, *J. Combin. Theory Ser. A* **49** (1988), 389–391.

P. Frankl, An extremal problem for two families of sets, *Europ. J. Combin.* **3** (1982), 125–127.

Z. Füredi, Geometrical solution of an intersection problem for two hypergraphs, *Europ. J. Combin.* **5** (1984), 133–136.

F. Jaeger and C. Payan, Nombre maximal d'arêtes d'un hypergraphe  $\tau$ -critique de rang  $h$ , *C. R. Acad. Sci. Paris Sér. A-B* **273** (1971), A221–A223.

G.O.H. Katona, Solution of a problem of A. Ehrenfeucht and J. Mycielski, *J. Combin. Theory Ser. A* **17** (1974), 265–266.

L. Lovász, Flats in matroids and geometric graphs, in *Combinatorial Surveys*, P.J. Cameron (ed.), Academic Press, (1977), pp. 45–86.

L. Lovász, Topological and algebraic methods in graph theory, in *Graph Theory and Related Topics*, J.A. Bondy and U.S.R. Murty (eds), Academic Press, (1979), pp. 1–15.

- J. Talbot, A new Bollobás-type inequality and applications to  $t$ -intersecting families of sets, *Discrete Math.* **285** (2004), 349–353.
- Z. Tuza, Helly-type hypergraphs and Sperner families, *Europ. J. Combin.* **5** (1984), 185–187.
- T.G. Tarján, Complexity of lattice-configurations, *Studia Sci. Math. Hungar.* **10** (1975), 203–211.
- C.Z. Zhu, On two set-systems with restricted cross-intersections, *Europ. J. Combin.* **16** (1995), 655–658.

## 105. Saturated Hypergraphs

The minimal number of edges in a  $k$ -saturated  $r$ -uniform hypergraph with  $n$  vertices is

$$\binom{n}{r} - \binom{n-k+r}{r}.$$

*Proof.* To see that the minimum is at most this much, let  $H_0$  be the  $r$ -uniform hypergraph with vertex set  $[n]$  in which the edge set is the set of  $r$ -subsets of  $[n]$  meeting  $[k-r]$ . Clearly,  $H_0$  is  $k$ -saturated, and has  $\binom{n}{r} - \binom{n-k+r}{r}$  edges.

Let us turn to the main assertion that the minimum is at least this much. Let  $C_1, C_2, \dots, C_m$  be the missing hyperedges, i.e., the  $r$ -subsets of  $V$  that are not edges of  $H$ . Then for every  $C_i$  there is a set  $D_i \subset V \setminus C_i$  of  $k-r$  vertices such that, with the exception of  $C_i$ , all  $r$ -subsets of  $C_i \cup D_i$  are edges of  $H$ . In particular,  $C_j \not\subset C_i \cup D_i$  for every  $i$  and  $j$ . Hence, by inequality (38) in Problem 104,

$$\sum_{i=1}^m \binom{n-|D_i|}{|C_i|}^{-1} = m \binom{n-k+r}{r}^{-1} \leq 1,$$

i.e.,

$$m \leq \binom{n-k+r}{r},$$

as claimed. □

**Notes.** This was the original (and rather trivial) application of Bollobás's Lemma from Problem 104.

## 106. The Norm of Averages: Hardy's Inequality

For every  $p > 1$ , there is a constant  $C_p$  (depending only on  $p$ ) such that the following assertion holds.

Let  $x = (x_k)_1^n$  be a sequence of non-negative reals, and let  $a = a(x) = (a_k)_1^n$  be the sequence of its averages, so that  $a_k = A_k/k$ , where  $A_k = \sum_{i=1}^k x_i$ . Then

$$\|a\|_p \leq C_p \|x\|_p,$$

where  $\|\cdot\|_p$  is the  $p$ -norm, i.e.,  $\|u\|_p^p = \sum_k |u_k|^p$ .



Fig. 79. G.H. Hardy.

*First Proof.* We shall show that  $C_p = p \zeta(p)$  will do, where  $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$  is the zeta function.

Note that, with  $A_0$  defined to be 0, we have  $A_k - A_{k-1} = x_k$ , and so the convexity of  $t^p$  for  $p > 1$  and  $t > 0$  implies that

$$A_k^p - A_{k-1}^p \leq p x_k A_k^{p-1},$$

with equality if and only if  $A_k - A_{k-1}$ , i.e.,  $x_k = 0$ .

For  $k = 1, 2, \dots$ , set  $\Phi_k = k^{-p} + (k+1)^{-p} + \dots$ . Then

$$\begin{aligned} \|a\|_p^p &= \sum_{k=1}^n (A_k/k)^p = \sum_{k=1}^n A_k^p (\Phi_k - \Phi_{k+1}) \\ &= \sum_{k=1}^n (A_k^p - A_{k-1}^p) \Phi_k - A_n^p \Phi_{n+1} \\ &\leq \sum_{k=1}^n (A_k^p - A_{k-1}^p) \Phi_k \leq p \sum_{k=1}^n x_k A_k^{p-1} \Phi_k. \end{aligned}$$

But

$$\begin{aligned} \Phi_k &= k^{-p} + (k+1)^{-p} + \dots + (2k)^{-p} + \dots + (3k)^{-p} + \dots \\ &< k(k^{-p} + (2k)^{-p} + \dots) = k^{1-p} \zeta(p), \end{aligned}$$

so

$$\|a\|_p^p \leq p \sum_{k=1}^n x_k A_k^{p-1} k^{1-p} \zeta(p) = p \zeta(p) \sum_{k=1}^n x_k a_k^{p-1}.$$

Hölder's inequality, applied with the conjugate exponents  $p$  and  $p/(p-1)$ , tells us that if  $b_k$  and  $c_k$  are non-negative then

$$\sum b_k c_k^{p-1} \leq \left( \sum b_k^p \right)^{1/p} \left( \sum c_k^p \right)^{(p-1)/p}. \quad (39)$$

Hence,

$$\|a\|_p^p \leq p \zeta(p) \left( \sum x_k^p \right)^{1/p} \left( \sum a_k^p \right)^{(p-1)/p},$$

and so

$$\|a\|_p \leq p \zeta(p) \|x\|_p.$$

□

*Second Proof.* We may suppose that  $n \geq 2$  and  $x_k > 0$  for every  $k$ , since otherwise there is nothing to prove. As before, we set  $a_0 = 0$ , so that  $x_k = ka_k - (k-1)a_{k-1}$  for  $k = 1, \dots, n$ . Our aim is to show that

$$\|a\|_p < \frac{p}{p-1} \|x\|_p, \quad (40)$$

so  $C_p = \frac{p}{p-1}$  will do.

Setting  $S = \sum_{k=1}^n x_k^p$ ,  $U = \sum_{k=1}^n a_k^p$ , and  $V = \sum_{k=1}^n x_k a_k^{p-1}$ , our task is to show that

$$U < \left( \frac{p}{p-1} \right)^p S.$$

To this end, note that if  $0 \leq u \leq v$  then

$$pu^{p-1} - pv^{p-1} \leq 0,$$

implying that

$$u^p + (p-1)v^p - puv^{p-1} \geq 0.$$

Therefore,

$$\begin{aligned} a_k^p - \frac{p}{p-1} x_k a_k^{p-1} &= a_k^p - \frac{p}{p-1} (ka_k - (k-1)a_{k-1}) a_k^{p-1} \\ &= \left(1 - \frac{kp}{p-1}\right) a_k^p + \frac{k-1}{p-1} p a_{k-1} a_k^{p-1} \\ &\leq \left(1 - \frac{kp}{p-1}\right) a_k^p + \frac{k-1}{p-1} ((p-1)a_k^p + a_{k-1}^p) \\ &= \frac{1}{p-1} ((k-1)a_{k-1}^p - ka_k^p). \end{aligned}$$

Summing these inequalities, we find that

$$U - \frac{p}{p-1} V \leq -\frac{na_n^p}{p-1} < 0,$$

and so

$$U < \frac{p}{p-1} V.$$

Also, exactly as in the first proof, applying Hölder's inequality in the form (39), we see that

$$V = \sum_{k=1}^n x_k a_k^{p-1} \leq \left(\sum_{k=1}^n x_k^p\right)^{1/p} \left(\sum_{k=1}^n a_k^p\right)^{(p-1)/p} = S^{1/p} U^{(p-1)/p}.$$

Therefore,

$$U < \frac{p}{p-1} V \leq \frac{p}{p-1} S^{1/p} U^{(p-1)/p},$$

and so

$$U < \left(\frac{p}{p-1}\right)^p S,$$

as desired. □

**Notes.** For Hardy, the precursor of his inequality was Hilbert's famous inequality that a real series

$$\sum \frac{x_n x_m}{n+m}$$

is convergent whenever  $\sum x_n^2$  is convergent. Hilbert's proof was based on considerations drawn from the theory of Fourier series; other proofs were given by Wiener and Schur. Hardy kept trying to find a proof 'by arguments of an entirely simple and elementary kind', and eventually found one, whose method, he felt, lacked 'nothing in directness or simplicity'. Nevertheless, he soon returned to the topic to give yet another proof, which was further simplified by Marcel Riesz. This is the first proof presented above, giving the constant  $C_p = p\zeta(p)$ .

Landau (in a letter to Schur) and Elliott (whose beautiful proof is given in the second solution) proved the inequality with  $C_p = \frac{p}{p-1}$ . As pointed out by Landau, taking  $x_k = 1/k^{p+\varepsilon}$ ,  $0 < \varepsilon < p - 1$ , one can show that  $C_p = \frac{p}{p-1}$  is the optimal constant. Returning to the problem in 1925, Hardy also gave an integral inequality analogue of his inequality, together with some extensions.

E.B. Elliott, A simple exposition of some recently proved facts as to convergency, *J. London Math. Soc.* **1** (1926), 93–96.

G.H. Hardy, Notes on some points in the integral calculus, *Messenger of Mathematics* **48** (1919), 107–112.

G.H. Hardy, Note on a theorem of Hilbert, *Mathematische Zeitschrift* **6** (1920), 314–317.

G.H. Hardy, Notes on some points in the integral calculus, *Messenger of Mathematics* **54** (1925), 150–156.

E. Landau, A note on a theorem concerning series of positive terms: extract from a letter of Prof. E. Landau to Prof. I. Schur (communicated by G.H. Hardy), *J. London Math. Soc.* **1** (1926), 38–39.



## 107. The Average of Geometric Means: Carleman's Inequality

Let  $b_1, \dots, b_n$  be positive numbers. Then

$$\sum_{k=1}^n (b_1 \cdots b_k)^{1/k} \leq e \sum_{k=1}^n b_k.$$

*Proof.* In the second solution of Problem 106, we proved Hardy's inequality with the best constant:

$$\sum_{k=1}^n a_k^p \leq \left( \frac{p}{p-1} \right)^p \sum_{k=1}^n x_k^p,$$

whenever  $x_1, \dots, x_n$  are positive numbers, and  $a_k = (x_1 + \cdots + x_k)/k$  is the average of the first  $k$  of them. This inequality is our starting point.

For positive reals  $c_1, \dots, c_n$ , we have

$$\lim_{p \rightarrow \infty} \left( \sum_{i=1}^k c_i^{1/p} / k \right)^p = (c_1 \cdots c_k)^{1/k}.$$

Hence, setting  $x_i = b_i^{1/p}$ ,

$$a_k^p = \left( \sum_{i=1}^k x_i / k \right)^p = \left( \sum_{i=1}^k b_i^{1/p} / k \right)^p \rightarrow (b_1 \cdots b_k)^{1/k}.$$

Therefore,  $\lim_{p \rightarrow \infty} (p/(p-1))^p = e$  implies that

$$\sum_{k=1}^n (b_1 \cdots b_k)^{1/k} \leq e \sum_{k=1}^n b_k,$$

as required. □

**Notes.** As remarked by Hardy in his 1925 paper, it was George Pólya who observed that Hardy's inequality with the best constant implies Carleman's inequality in this question. Pólya also noted that another consequence of Hardy's inequality is that

$$\int_0^\infty \exp \left\{ \frac{1}{x} \int_0^x f(t) dt \right\} dx \leq \int_0^\infty \exp f(x) dx.$$

T. Carleman, Sur les fonctions quasi-analytique, *Conférences faites au cinquième congrès des mathématiciens Scandinaves*, Helsingfors, (1923), pp. 181–196.

G.H. Hardy, Notes on some points in the integral calculus, *Messenger of Mathematics* **54** (1925), 150–156.



## 108. Triangulating Squares

*A square cannot be cut into an odd number of triangles of the same area.*

*Proof.* First, let us see why the *Hint* may make sense, i.e., how 2-adic norms could be of use in proving that every tessellation consists of an even number of triangles. We may and shall assume that our square is the unit square  $U$  with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(1, 1)$  and  $(0, 1)$  in the plane  $\mathbb{R}^2$ ; thus the  $U$  is cut into  $n$  triangles, each of area  $1/n$ . Now, the area of a triangle is given by the cross-product of the sides: the triangle with vertices  $(0, 0)$ ,  $(x_1, y_1)$  and  $(x_2, y_2)$  has area  $a = \frac{1}{2}|x_1y_2 - x_2y_1|$ . Our task is to prove that if the area of one of our triangles is of the form  $a = 1/n$  then  $n$  is even. But this holds if and only if the 2-adic norm of  $a$  is at least 2. Our aim then is to find a triangle of our tessellation whose form does guarantee that the 2-adic norm of its area is greater than 1. In fact, we shall prove the claim below, which is considerably more than we need.

**Claim.** *Let  $T$  be a tessellation of the unit square into finitely many triangles. Then one of the triangles is such that the 2-adic norm of its area is at least 2.*

Before we embark on our proof of this Claim, let us recall some basic facts concerning the 2-adic norm, at least on *rational numbers*. The 2-adic norm  $\|x\|$  of a non-zero rational number  $x = 2^k \frac{a}{b}$  with  $a$  and  $b$  odd integers is  $\|x\| = 2^{-k}$ , and the norm of 0 is 0. It is easily seen that on the rational numbers the 2-adic norm has the following properties.

- (i) The norm of a non-zero rational is strictly positive.
- (ii) The norm of a product is the product of the norms.
- (iii) If  $\|x\| > \|y\|$  then  $\|x + y\| = \|x\|$ , and if  $\|x\| = \|y\| > 0$  then  $\|x + y\| \leq \|x\|$ .

Indeed, (i) and (ii) are immediate from the definition, and (iii) needs only a little checking. Thus, let  $x = 2^k \frac{a}{b}$  and  $y = 2^\ell \frac{c}{d}$ , with  $a, b, c, d$  odd. If  $k < \ell$ , then  $x + y = 2^k \frac{ad + 2^{\ell-k}bc}{bd}$ , with  $ad + 2^{\ell-k}bc$  and  $bd$  odd; if  $k = \ell$  then  $x + y = 2^{k+1} \frac{(ad+bc)/2}{bd}$ , with  $bd$  odd and  $(ad + bc)/2$  an integer.

It is also true, although far from obvious, that the 2-adic norm can be extended to the reals in such a way that (i), (ii) and (iii) remain true. We shall make use of this, keeping the notation  $\|x\|$  for the 2-adic norm of a real number  $x$ .

Let us 'colour' the points of the plane  $\mathbb{R}^2$  in a way that will enable us to locate a suitable triangle in a tessellation of the unit square  $U$ . Given a point  $x = (x_1, x_2) \in \mathbb{R}^2$ , colour it 0 if both of the 2-adic norms  $\|x_1\|$  and  $\|x_2\|$  are less than 1, colour it

1 if  $\|x_1\| \geq 1$  and  $\|x_1\| \geq \|x_2\|$ , and colour it 2 otherwise, i.e., if  $\|x_2\| \geq 1$  and  $\|x_2\| > \|x_1\|$ . Intuitively, a point is coloured 0 if both its coordinates are ‘small’ (in particular, the origin has colour 0); the colour is 1 if the *first* coordinate is ‘large’ (has norm at least 1) and is no smaller than the second; finally, the colour is 2 if the *second* coordinate is ‘large’ and larger than the first. We write  $c(x)$  for the colour of a point  $x$ .

Property (iii) implies that if  $c(t) = 0$  then  $c(x + t) = c(x)$  for every  $x$ : translation by a vector of colour 0 does not change the colour of a point.

More importantly, let us note that if  $T$  is a *rainbow-coloured* triangle (i.e., a triangle whose vertices are coloured 0, 1 and 2) then the 2-adic norm of the area of  $T$  is at least 2. In particular, as three points on a line form a degenerate triangle, a triangle of area 0, no line contains points of all three colours. Since translation by a vector of colour 0 changes neither the colours of the vertices, nor the area of the triangle (and so, *a fortiori*, the 2-norm of the area does not change either), in proving this we may assume that the triangle  $T$  is  $0xy$  with  $c(x) = 1$  and  $c(y) = 2$ . If  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$  then  $\|x_1\| \geq \|x_2\|$ ,  $\|x_1\| \geq 1$ ,  $\|y_2\| > \|y_1\|$  and  $\|y_2\| \geq 1$ , so  $\|x_1y_2\| = \|x_1\|\|y_2\| > \|x_2\|\|y_1\| = \|x_2y_1\|$  and  $\|x_1y_2\| \geq 1$ . Therefore, properties (ii) and (iii) of the 2-adic norm imply that

$$\|\text{area}(T)\| = \left\| \frac{1}{2}(x_1y_2 - x_2y_1) \right\| = 2\|x_1y_2 - x_2y_1\| = 2\|x_1y_2\| \geq 2,$$

as asserted. Consequently, the Claim follows (and so does our proof) if we show that any tessellation  $\mathcal{T}$  of the unit square  $U$  into finitely many triangles contains a rainbow-coloured triangle.

Suppose, for a contradiction, that  $\mathcal{T}$  does not contain a rainbow-coloured triangle. Let  $G$  be the graph of the tessellation: the vertices of  $G$  are the vertices of the triangles, and two vertices  $a$  and  $b$  are joined by an edge if they are on a side of a triangle and the (straight line) segment  $ab$  does not contain another vertex. Thus the tessellation is just a straight line drawing of the plane graph  $G$ .

Colour an edge *black* if its vertices are coloured 0 and 2. A simple consequence of the fact that no line contains points of all three colours is that a 02-segment (a straight line segment from a vertex of colour 0 to a vertex of colour 2) contains an odd number of black edges (see Figure 80). Consequently, a 00-segment contains an even number of black edges, and so does a 22-segment. Furthermore, no other segment (from a vertex to a vertex) contains any black edges.

Write  $A$  for the total number of black edges inside the square  $U$  and  $B$  for the total number of black edges on the boundary. Since the square  $U$  has only one 02-side,  $B$  is odd. Consequently,  $2A + B$ , the total number of black edges on the sides

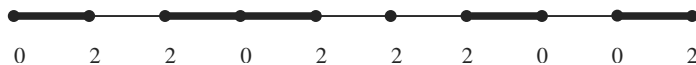


Fig. 80. A 02-segment divided by some vertices of colour 0 and 2, with the black edges drawn thick. Note that, going from left to right, the first black edge goes from a vertex of colour 0 to one of colour 2, the second from 2 to 0, the third from 0 to 2, and so on. As the last goes from colour 0 to colour 2, there are an odd number of black edges.

of the triangles in  $\mathcal{T}$ , is odd. Therefore, at least one of the triangles of the tessellation  $\mathcal{T}$  has an odd number of black edges on its side. However, this is impossible, as only a rainbow-coloured triangle may have an odd number of black edges on its sides, since only a rainbow-coloured triangle has an odd number of 02-edges.

□

B. Bekker, S. Vostokov and Yu. Ionin, 2-adic numbers (in Russian), *Kvant* (1979), 26–31]. For a translation, see *Kvant Selecta: Algebra and Analysis, I*, pp. 99–109, Math. World **14**, Amer. Math. Soc., (1999).

J.W.S. Cassels, *Local Fields*, London Math. Soc. Student Texts **3**, Cambridge University Press, (1986). xiv + 360 pp.

## 109. Strongly Separating Families

*The minimal number of sets in a family of sets separating  $n$  elements is precisely the minimal  $t$  such that*

$$n \leq \binom{t}{\lfloor t/2 \rfloor}.$$

*In particular, the minimum is about  $\log_2 n$ .*

*Proof.* Let  $\mathcal{S} = \{S_1, S_2, \dots, S_t\}$  be a family of subsets of  $X = [n]$ . For  $i \in [n]$ , set  $A_i = \{k : i \in S_k\}$ , so that  $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$  is a family of subsets of the  $t$ -element set  $T = [t]$ . Clearly, not only does  $\mathcal{S}$  determine  $\mathcal{A}$ , as we have done it, but  $\mathcal{A}$  determines  $\mathcal{S}$  as well.

We claim that  $\mathcal{S}$  strongly separates the elements of  $X$  if and only if  $\mathcal{A}$  is a Sperner system, i.e., no set  $A_i$  is contained in any other  $A_j$ . Once we realize that this claim may be true, checking it is a triviality: indeed,  $A_i$  is not contained in  $A_j$  if, and only if some  $S_k$  contains  $i$ , but not  $j$ . Since the maximal size of a Sperner system on  $t$  elements is  $\binom{t}{\lfloor t/2 \rfloor}$ , we are done.  $\square$

**Notes.** The problem is due to Dickson, who showed that the minimum is about  $\log_2 n$ , and the exact bound above was given by Spencer.

Dickson, T. J., On a problem concerning separating systems of a finite set, *J. Combin. Theory* **7** (1969), 191–196.

Spencer, J., Minimal completely separating systems, *J. Combin. Theory* **8** (1970), 446–447.

## 110. Strongly Separating Systems of Pairs of Sets

Let  $T(n)$  be the minimum of  $\sum_{i=1}^N |S_i \cup T_i|$  subject to the condition that  $(S_i, T_i)_{i=1}^N$  is a strongly separating system on  $[n]$ , where  $n \geq 2$ . Furthermore, let  $t$  be the minimal natural number such that

$$\binom{t+1}{\lfloor (t+1)/2 \rfloor} > n.$$

Then  $T(n) \geq nt$ , with equality if and only if  $n = \binom{t}{\lfloor t/2 \rfloor}$ . In particular,  $T(n) = (\log_2 n + \frac{1}{2} \log_2 \log n + O(1))n$ .

*Proof.* Let  $(S_1, T_1), \dots, (S_N, T_N)$  be a strongly separating system on  $[n]$ . For  $1 \leq j \leq n$ , define

$$\begin{aligned} A_j &= \{i : j \in S_i\}, \\ B_j &= \{i : j \in T_i\}. \end{aligned}$$

Then  $\{(A_j, B_j) : 1 \leq j \leq n\}$  is a cross-intersecting family. Indeed,  $A_j \cap B_j = \emptyset$  since  $S_i$  and  $T_i$  are disjoint; also, if  $j \neq h$  then  $j \in S_i$  and  $h \in T_i$  for some  $i$ , so  $A_j \cap B_h \neq \emptyset$ . Hence, by Bollobás's lemma in Problem 104,

$$\sum_{j=1}^n \binom{|A_j| + |B_j|}{|A_j|}^{-1} \leq 1. \quad (41)$$

We shall make use of this inequality in conjunction with the simple fact that if  $1 \leq a \leq b-2$  then

$$\binom{a}{\lfloor a/2 \rfloor}^{-1} + \binom{b}{\lfloor b/2 \rfloor}^{-1} \geq \binom{a+1}{\lfloor (a+1)/2 \rfloor}^{-1} + \binom{b-1}{\lfloor (b-1)/2 \rfloor}^{-1}. \quad (42)$$

Now

$$\sum_{i=1}^N |S_i \cup T_i| = \sum_{i=1}^n |A_i \cup B_i|.$$

By (41) this is at least

$$\min \left\{ \sum_{i=1}^n (a_i + b_i) : \sum_{i=1}^n \binom{a_i + b_i}{a_i}^{-1} \leq 1 \right\},$$

which is at least

$$\min \left\{ \sum_{i=1}^n c_i : \sum_{i=1}^n \binom{c_i}{\lfloor c_i/2 \rfloor}^{-1} \leq 1 \right\},$$

where the minimum is taken over all sequences  $c_1, \dots, c_n$  of positive integers.

Consider a sequence  $c_1, \dots, c_n$  that achieves this minimum and (subject to this) has  $\sum c_i^2$  minimal. It follows from (42) that the  $c_i$  take at most two values, say  $t$  and  $t+1$  (where  $t = \min c_i$ ). We have

$$\binom{t}{\lfloor t/2 \rfloor} \leq n < \binom{t+1}{\lfloor (t+1)/2 \rfloor}$$

and so  $\sum_{i=1}^n c_i \geq tn$ , with equality only when  $n = \binom{t}{\lfloor t/2 \rfloor}$ . Note that, in this case, equality is achieved by starting with the cross-intersecting family  $\{(A, [t] \setminus A) : A \in [t]^{\lfloor t/2 \rfloor}\}$ , which, as we have seen, gives a strongly separating family with  $t$  pairs on  $[n]$  with  $n = \binom{t}{\lfloor t/2 \rfloor}$ .  $\square$

**Notes.** The study of separating systems was started by Rényi in 1961. Clearly, strong separation on  $[n]$  amounts to covering the complete directed graph  $\overrightarrow{K_n}$  with complete bipartite graphs, with all edges going from one class to another;  $T(n)$  is the minimum of the sum of orders of the covering bipartite graphs. The result and the proof above are due to Bollobás and Scott. Separating systems  $(S_i, T_i)$  with restrictions on the cardinalities  $|S_i|, |T_i|$  have been studied by numerous people; for some of this work, see the references below.

B. Bollobás and A. Scott, On separating systems, *Europ. J. Combin.*, to appear

G. Katona, On separating systems of a finite set. *J. Combin. Theory* **1** (1966), 174–194.

A. Kündgen, D. Mubayi and P. Tetali, Minimal completely separating systems of  $k$ -sets, *J. Combin. Theory Ser. A* **93** (2001), 192–198.

C. Ramsay and I.T. Roberts, Minimal completely separating systems of sets, *Australas. J. Combin.* **13** (1996), 129–150.

I. Wegener, On separating systems whose elements are sets of at most  $k$  elements, *Discrete Math.* **28** (1979), 219–222.



## 111. The Maximum Edge-Boundary of a Down-set

The edge-boundary of a down-set in the cube  $Q^n = \{0, 1\}^n$  has at most  $\lceil n/2 \rceil \binom{n}{\lfloor n/2 \rfloor}$  edges.

*Proof.* We shall prove more than asserted in the exercise: for every  $m$ ,  $0 \leq m \leq 2^n$ , we shall determine the maximum edge-boundary of a down-set with  $m$  elements.

First, let us recall some basic facts about the  $n$ -dimensional cube  $Q^n$ . This is a graph with  $2^n$  vertices and  $n2^{n-1}$  edges: the vertex set is

$$Q_n = \{0, 1\}^n = \{\mathbf{x} = (x_1, x_2, \dots, x_n) : x_i = 0 \text{ or } 1\},$$

and two vertices are joined by an edge if they differ in precisely one coordinate.

The cube  $Q^n$  is naturally identified with the power set  $\mathcal{P}(n)$  of  $[n]$ : this identification is given by the correspondence  $\mathbf{a} = (a_1, \dots, a_n) \leftrightarrow A = \{i : a_i = 1\}$ . Under this identification  $\mathcal{P}(n)$  is also a graph with  $2^n$  vertices: the vertices are all  $2^n$  subsets of  $[n] = \{1, 2, \dots, n\}$ , and two sets  $A, B \in \mathcal{P}(n)$  are joined by an edge when they differ in precisely one element. With a slight abuse of notation, we freely interchange  $Q^n$  and  $\mathcal{P}(n)$ , and use the same notation for the graphs as for their vertex sets:  $Q^n = \{0, 1\}^n = \mathcal{P}(n)$ .

A set  $D \subset Q^n$  is a *down-set* if  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in D$  implies that  $\mathbf{y} = (y_1, y_2, \dots, y_n) \in D$  whenever  $y_i \leq x_i$  for every  $i$ . An *up-set* is defined similarly: it is also the complement of a down-set. For  $\mathcal{P}(n)$  these notions are perhaps even more natural: a set system  $\mathcal{D} \subset \mathcal{P}(n)$  is a *down-set* if  $A \in \mathcal{D}$  implies that  $B \in \mathcal{D}$  whenever  $B \subset A$ . Similarly,  $\mathcal{U} \subset \mathcal{P}(n)$  is an *up-set* if  $A \in \mathcal{U}$  implies that  $B \in \mathcal{U}$  whenever  $B \supset A$ .

We shall write  $b(S)$  for the size of the edge-boundary of a set  $S \subset Q^n$ , i.e., for the number of edges joining a vertex in  $S$  to a vertex not in  $S$ , and  $e(S)$  for the number of edges spanned by  $S$ . Then our task is to show that

$$b(D) \leq \lceil n/2 \rceil \binom{n}{\lfloor n/2 \rfloor}$$

for every down-set  $D \subset Q^n$ . It will be slightly more natural to prove this using the terminology of set systems: this is what we shall do.

Since  $\mathcal{P}(n)$  is an  $n$ -regular graph, if a vertex  $A \in \mathcal{A}$  is incident with  $b$  boundary edges then it is adjacent with  $n - b$  edges spanned by  $A$ , so

$$b(\mathcal{A}) + 2e(\mathcal{A}) = n|\mathcal{A}|. \quad (43)$$

Hence, given the cardinality of  $\mathcal{A}$ , the edge-boundary  $b(\mathcal{A})$  is maximized if  $e(\mathcal{B})$  is minimized.

Now, for a down-set  $\mathcal{D}$ , counting each edge  $AB$  from the larger of the two, we see that

$$e(\mathcal{D}) = \sum_{D \in \mathcal{D}} |D|, \quad (44)$$

since in  $\mathcal{D}$  every set  $D \in \mathcal{D}$  is joined to all  $|D|$  sets  $D'$  with  $D' \subset D$  and  $|D \setminus D'| = 1$ .

As promised, we shall prove considerably more than asked by the question. In fact, everything we want follows easily from the identities (43) and (44). Let

$$m = \sum_{k=0}^{r-1} \binom{n}{k} + s \leq \sum_{k=0}^r \binom{n}{k}, \quad (45)$$

and set

$$f(m) = (n - r + 1) \binom{n}{r-1} + s(n - 2r). \quad (46)$$

We shall show that if  $\mathcal{D} \subset \mathcal{P}(n)$  is a down-set with  $|\mathcal{D}| = m$  then

$$b(\mathcal{D}) \leq f(m).$$

Since

$$\max_m f(m) = \max_r (n - r) \binom{n}{r} = \lceil n/2 \rceil \binom{n}{\lfloor n/2 \rfloor},$$

this does solve our problem.

As there are  $\binom{n}{k}$  sets of size  $k$  (and so contributing ‘only’  $k$  to the sum in (44)), for a down-set  $\mathcal{D}$  of cardinality  $m$  as above,  $e(\mathcal{D})$  is minimized if  $\mathcal{D}$  is the union of some full layers: one empty set,  $\binom{n}{1}$  singletons,  $\binom{n}{2}$  sets of two elements, and so on, ending with  $\binom{n}{r-1}$  sets of  $r-1$  elements and  $s \leq \binom{n}{r}$  sets of  $r$  elements. Exactly the same distribution of sets maximizes  $b(\mathcal{D})$ , showing that  $b(\mathcal{D}) \leq f(m)$ , as claimed.  $\square$

## 112. Partitioning a Subset of the Cube

For every subset  $\mathcal{A}$  of the cube  $Q^n$  there are two parallel faces such that the difference of the numbers of elements of  $\mathcal{A}$  on these faces is at most

$$\frac{\lceil n/2 \rceil}{n} \binom{n}{\lfloor n/2 \rfloor} \leq 2^n / \sqrt{n}.$$

Furthermore, if  $\varepsilon > 0$  and  $\eta(n) \rightarrow 0$  then if  $n$  is large enough and  $\mathcal{A} \subset Q^n$  has at least  $2^{(1-\eta(n))n}$  vertices, then for some two parallel faces the difference is at most  $\varepsilon|\mathcal{A}|$ .

*Proof.* Let us make the problem more formal. For  $1 \leq i \leq n$ , set  $d_i(\mathcal{A}) = \|\mathcal{A}_-^i\| - \|\mathcal{A}_+^i\|$ , where

$$\mathcal{A}_-^i = \{A \in \mathcal{A} : i \notin A\}$$

and

$$\mathcal{A}_+^i = \{A \in \mathcal{A} : i \in A\},$$

i.e.,  $(\mathcal{A}_-^i, \mathcal{A}_+^i)$  is the partition of  $\mathcal{A} \subset Q^n$  given by the two faces in direction  $i$ . Then our first task is to prove that

$$d_i(\mathcal{A}) \leq \frac{\lceil n/2 \rceil}{n} \binom{n}{\lfloor n/2 \rfloor} \quad (47)$$

for some  $i$ . The second task is to show that for some  $i$  the difference  $d_i(\mathcal{A})$  is at most  $\varepsilon|\mathcal{A}|$ .

Following the *Hint*, first we show that in proving the existence of a small  $d_i(\mathcal{A})$  we may assume that  $\mathcal{A}$  is a down-set. Although this could be accomplished in an informal way, we shall be rather formal (and a little heavy-handed), as we wish to use these ideas in later proofs as well.

To turn  $\mathcal{A}$  into a down-set, we apply *compressions*: picking an element ('direction')  $j$ , we delete  $j$  from every set  $A \in \mathcal{A}$ , provided this deletion does not decrease the number of sets in  $\mathcal{A}$ . The new set system is  $C_j(\mathcal{A})$ , the *down-compression of  $\mathcal{A}$  in the direction  $i$* . Formally, for  $A \in \mathcal{A}$  set

$$C_{j,\mathcal{A}}(A) = \begin{cases} A \setminus \{j\} & \text{if } j \in A \text{ and } A \setminus \{j\} \notin \mathcal{A}, \\ A & \text{otherwise,} \end{cases}$$

and let

$$C_j(\mathcal{A}) = \{C_{j,\mathcal{A}}(A) : A \in \mathcal{A}\}.$$

Clearly, the set system  $C_j(\mathcal{A})$  has precisely as many elements as  $\mathcal{A}$ , and  $\|C_j(\mathcal{A})\| < \|\mathcal{A}\|$  unless  $C_j(\mathcal{A}) = \mathcal{A}$ . Also,  $\mathcal{A}$  is a down-set if and only if  $C_j(\mathcal{A}) = \mathcal{A}$  for every  $j$ .

The definition of  $d_i(\mathcal{A})$  implies that  $d_i(\mathcal{A}) \leq d_i(C_i(\mathcal{A}))$  and  $d_i(\mathcal{A}) = d_i(C_j(\mathcal{A}))$  for every  $j \neq i$ . Consequently, in proving that (47) holds for some  $i$ , we may replace  $\mathcal{A}$  by  $C_j(\mathcal{A})$ . Successive applications of this down-compression give us a down-set that can replace  $\mathcal{A}$ . To make this precise, if  $\mathcal{A}$  is not a down-set, then  $\mathcal{A}' = C_j(\mathcal{A}) \neq \mathcal{A}$  for some  $j$ : replace  $\mathcal{A}$  by  $\mathcal{A}'$ .

Next, if  $\mathcal{A}'$  is not a down-set then  $\mathcal{A}'' = C_h(\mathcal{A}') \neq \mathcal{A}'$  for some  $h$ : replace  $\mathcal{A}'$  by  $\mathcal{A}''$ . Proceeding in this way, the sequence  $\mathcal{A}, \mathcal{A}', \mathcal{A}'', \dots$  has to stop after a finite number of steps, since  $\|\mathcal{A}\| > \|\mathcal{A}'\| > \|\mathcal{A}''\| > \dots$ . The final term of this sequence,  $\mathcal{A}^*$ , is a down-set with  $|\mathcal{A}^*| = |\mathcal{A}|$  such that  $d_i(\mathcal{A}^*) \geq d_i(\mathcal{A})$  for every  $i$ . Hence, in proving that (47) holds for some  $i$ , we may and shall assume that  $\mathcal{A}$  is a down-set.

Now, for a down-set  $\mathcal{A}$ , the difference  $d_i(\mathcal{A})$  is the number of edges in direction  $i$  in the edge-boundary of  $\mathcal{A}$ . Hence,  $\sum_{i=1}^n d_i(\mathcal{A})$  is precisely the cardinality of the edge-boundary  $\partial_e(\mathcal{A})$  of  $\mathcal{A}$ . By Problem 111, this cardinality is at most  $\lceil n/2 \rceil \binom{n}{\lfloor n/2 \rfloor}$ , so

$$d_i(\mathcal{A}) \leq \frac{\lceil n/2 \rceil}{n} \binom{n}{\lfloor n/2 \rfloor} \leq 2^n / \sqrt{n}$$

for some  $i$ , proving the first assertion.

The second assertion follows from the more detailed assertion in Problem 111 that a down-set  $\mathcal{D} \subset \mathcal{P}(n)$  with  $m$  elements as in (45) has at least  $f(m)$  edges in its boundary, where  $f(m)$  is given by (46).  $\square$

### 113. Weakly Cross-intersecting Pairs: Frankl's Theorem

Let  $(A_i, B_i)$ ,  $i = 1, \dots, m$ , be pairs of disjoint sets with  $|A_i| = a$  and  $|B_i| = b$  such that  $A_i \cap B_j \neq \emptyset$  for  $1 \leq i < j \leq m$ . Then  $m \leq \binom{a+b}{a}$ .

*Proof.* Let  $X = \bigcup_{i=1}^m (A_i \cup B_i)$  be our ground set. We may and shall assume that  $X$  is a set of points in  $V = \mathbb{R}^{a+1}$  such that  $X \cup \{0\}$  is in general position, i.e., such that no proper affine subspace of  $V$  contains  $a+2$  of the points of  $X \cup \{0\}$ . In particular, for every  $i$ , the  $a$  points in  $A_i$  and the origin span a subspace  $V_i$  of dimension  $a$ , and  $V_i$  contains no point of  $X \setminus A_i$ . Furthermore, let  $u_i$  be a unit normal of the hyperplane  $V_i$ . Thus, identifying a point  $v \in V$  with the vector  $v$ ,

$$V_i = \{v \in V : \langle v, u_i \rangle = 0\}.$$

Recall that a  $b$ -linear functional  $f : V^b = V \times \dots \times V \rightarrow \mathbb{R}$  is *symmetric* if  $f(v_1, \dots, v_b) = f(b_{\pi(1)}, \dots, v_{\pi(b)})$  for every permutation  $\pi \in S_b$ . The vector space of symmetric  $b$ -linear functionals on  $V$  has dimension  $\binom{a+b}{a}$ , as a basis of the space of symmetric  $b$ -linear functionals is identified with the collection of multisets of size  $b$  with elements from an  $(a+1)$ -set.

Now, for every set  $B_k = \{v_{k1}, \dots, v_{kb}\}$ , define

$$f_{B_k}(v_1, \dots, v_b) = \frac{1}{b!} \sum_{\pi \in S_b} \prod_{j=1}^b \langle v_{kj}, v_{\pi(j)} \rangle.$$

By definition, each  $f_{B_k}$  is a symmetric  $b$ -linear functional. Hence, our result follows if we show that the  $m$  functionals  $f_{B_k}$  are linearly independent.

Suppose that these functionals are not independent, i.e., there are constants  $c_1, \dots, c_m$ , not all zero, such that

$$f = \sum_{k=1}^m c_k f_{B_k}$$

is the zero functional. Let  $c_h$  be the first non-zero coefficient so that  $c_1 = \dots = c_{h-1} = 0$  and  $c_h \neq 0$ . We shall arrive at a contradiction by evaluating  $f$  on  $(u_h, \dots, u_h)$ . If  $1 \leq h < k \leq m$  then  $A_h \cap B_k \neq \emptyset$ , so  $v_{kj} \in A_h \subset V_h$  for some  $j$ ; therefore  $\langle v_{kj}, u_h \rangle = 0$ , implying that  $f_{B_k}(u_h, \dots, u_h) = 0$ . Furthermore, by our choice of  $h$ ,  $c_1 = \dots = c_{h-1} = 0$ , so

$$0 = f(u_h, \dots, u_h) = c_h f_{B_h}(u_h, \dots, u_h). \quad (48)$$

Finally, as  $A_h \cap B_h = \emptyset$  and the points are in general position,  $V_h \cap B_h = \emptyset$ ; therefore,  $\langle b_{hk}, u_h \rangle \neq 0$  for every  $b_{hk} \in B_h$ , implying that  $f_{B_h}(u_h, \dots, u_h) \neq 0$ , contradicting (48).  $\square$

**Notes.** Frankl's theorem above is a considerable extension of the special case of Bollobás' Lemma in which each  $A_i$  has  $a$  elements and each  $B_i$  has  $b$  elements. The beautiful and *a priori* rather surprising proof we have presented is that of Frankl; Kalai and Alon found similar (and just as beautiful) proofs as well.

Frankl's theorem implies the extension of the result in Problem 105 to weakly  $k$ -saturated uniform hypergraphs. An  $r$ -uniform hypergraph with vertex set  $[n]$  is *weakly  $k$ -saturated* if there is an enumeration  $E_1, \dots, E_m$  of the  $r$ -sets of  $[n]$  that are not edges such that if only  $E_1, \dots, E_i$  are added to the hypergraph then there is a complete  $k$ -graph containing  $E_i$  as an edge. Then, as conjectured by Bollobás,  $m \leq \binom{n-k+r}{r}$ , greatly strengthening the result in Problem 105.

N. Alon, On the density of sets of vectors, *Discrete Math.* **46** (1983), 199–202.

N. Alon, An extremal problem for sets with applications to graph theory, *J. Combin. Theory Ser. A* **40** (1985), 82–89.

B. Bollobás, Weakly  $k$ -saturated graphs, in *Beiträge zur Graphentheorie* (Kolloquium, Manebach, 1967), Teubner, Leipzig, (1968), pp. 25–31.

P. Frankl, An extremal problem for two families of sets, *Europ. J. Math.* **3** (1982), 125–127.

G. Kalai, Weakly saturated graphs are rigid, in *Convexity and Graph Theory*, M. Rosenfeld and J. Zaks (eds), *Ann. Discrete Math.* **20** (1984), 189–190.

## 114. Even Sets with Even Intersections

The maximal number of subsets of  $[n] = \{1, \dots, n\}$  such that each set has an even number of elements and any two sets intersect in an even number of elements is  $2^{\lfloor n/2 \rfloor}$ , i.e.,  $2^{n/2}$  if  $n$  is even and  $2^{(n-1)/2}$  if  $n$  is odd.

*Proof.* It is clear that we can have at least  $2^{\lfloor n/2 \rfloor}$  sets: indeed, pick  $\lfloor n/2 \rfloor$  disjoint pairs of elements, and consider all sets that can be formed by taking the union of some of them.

Our main task is then to prove that we cannot have more than  $2^{\lfloor n/2 \rfloor}$  sets. As suggested in the *Hint*, identify our sets with their characteristic vectors, and consider these vectors as elements of the  $n$ -dimensional vector space  $V = \mathbb{Z}_2^n$ , endowed with an inner product. Call a set  $A \subset V$  *self-orthogonal* if  $\langle \mathbf{a}, \mathbf{b} \rangle = 0$  for all  $\mathbf{a}, \mathbf{b} \in A$ , with  $\mathbf{a} = \mathbf{b}$  allowed.

Then what we have to prove is that a self-orthogonal set in the inner product space  $V$  consists of at most  $2^{\lfloor n/2 \rfloor}$  vectors.

As we shall see in a moment, this is immediate from the following basic result about inner product spaces: if  $U$  is a subspace of  $V$  then

$$\dim(U) + \dim(U^\perp) = \dim(V), \quad (49)$$

where  $U^\perp$  is the *orthogonal complement* of  $U$ :

$$U^\perp = \{\mathbf{v} \in V : \langle \mathbf{u}, \mathbf{v} \rangle = 0 \text{ for all } \mathbf{u} \in U\}.$$

To see (49), pick a basis  $(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k)$  of  $U$ , and extend it to a basis  $(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n)$  of  $V$ . Let  $(\mathbf{u}_1^*, \mathbf{u}_2^*, \dots, \mathbf{u}_n^*)$  be the dual basis, i.e., let  $u_j^* \in V$  be such that

$$\langle u_i, u_j^* \rangle = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Then it is clear that  $(\mathbf{u}_{k+1}^*, \mathbf{u}_{k+2}^*, \dots, \mathbf{u}_n^*)$  is a basis of  $U^\perp$ , since  $\mathbf{v} = \sum_{j=1}^n \lambda_j \mathbf{u}_j^*$  belongs to  $U^\perp$  if and only if  $\langle \mathbf{u}_i, \mathbf{v} \rangle = 0$  for every  $i$ ,  $1 \leq i \leq k$ , i.e.,  $\lambda_i = \langle \mathbf{u}_i, \mathbf{v} \rangle = 0$  for every  $i$ ,  $1 \leq i \leq k$ . Hence,  $\dim(U) = k$  implies that  $\dim(U^\perp) = n - k$ , proving (49).

And now for the proof. Let  $U$  be the subspace spanned by a self-orthogonal subset  $A$  of  $V$ , and set  $k = \dim(U)$ . Note that  $U$  is also self-orthogonal, because if  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in A$  then  $\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle = 0$  and  $\langle \mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{b} \rangle = 0$ . Since a self-orthogonal subspace is contained in its orthogonal complement, we have  $U \subset U^\perp$ , and so  $k \leq n - k$ , i.e.,  $k \leq \lfloor n/2 \rfloor$ . Hence  $A$  contains at most as many vectors as an  $\lfloor n/2 \rfloor$ -dimensional subspace, i.e., no more than  $2^{\lfloor n/2 \rfloor}$ .  $\square$

**Notes.** A slightly different way of defining  $U$  above is to say that we may assume that  $A$  is a *maximal* self-orthogonal subset. But then  $A$  itself is a subspace for the reason given above: if  $\mathbf{a}$  and  $\mathbf{b}$  are in  $A$  then  $\mathbf{a} + \mathbf{b}$  is orthogonal to  $A$ .

E.R. Berlekamp, On subsets with intersections of even cardinality, *Canad. Math. Bull.* **12** (1969), 471–474.

J.E. Graver, Boolean designs and self-dual matroids, *Linear Algebra and Appl.* **10** (1975), 111–128.



## 115. Sets with Even Intersections

For  $n \geq 6$ , the maximal number of subsets of  $[n] = \{1, \dots, n\}$  such that any two of them intersect in an even number of elements is  $2^{n/2}$  for  $n$  even and  $2^{\lfloor n/2 \rfloor} + 1 = 2^{(n-1)/2} + 1$  for  $n$  odd.

*Proof.* First, let us note what happens for small values of  $n$ . The system consisting of the empty set and the  $n$  singletons has the required property, so the maximum is at least  $n + 1$ . It is easy to check that for  $1 \leq n \leq 5$  we cannot do better, so the maximum is  $n + 1$ .

Now, let us turn to the proof of our assertion. That we can have as many sets as claimed is again trivial; indeed, for  $n$  even partition the ground set  $[n]$  into  $n/2$  pairs, and take all  $2^{n/2}$  unions of these pairs; for  $n$  odd, partition the even set  $[n-1]$  into  $(n-1)/2$  pairs, and take all  $2^{(n-1)/2}$  unions of these pairs, together with the singleton  $\{n\}$ .

All we have to do is dot the i's in the *Hint*. Let then  $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$  be a collection of subsets of  $[n]$ , with  $\mathcal{A}_0$  consisting of even sets and  $\mathcal{A}_1$  of odd sets, such that any two distinct elements of  $\mathcal{A}$  intersect in an even number of elements. Suppose  $\mathcal{A}_0$  consists of  $k$  sets and  $\mathcal{A}_1$  of  $\ell$  sets. We claim that

$$k \leq 2^{\lfloor (n-\ell)/2 \rfloor}. \quad (50)$$

From (50) it is only a slight nuisance to deduce our answer; we do not bore the reader with this deduction.

In order to prove (50) set  $\mathcal{A}_1 = \{A_1, A_2, \dots, A_\ell\}$ . For  $1 \leq i \leq \ell$ , define  $A_i^* = A_i \cup \{n+i\}$ . By construction,  $\mathcal{A}_1^* = \{A_1^*, A_2^*, \dots, A_\ell^*\}$  consists of even sets and so the family  $\mathcal{A}^* = \mathcal{A}_0 \cup \mathcal{A}_1^*$  of even subsets of  $[n+\ell]$  is such that any two of its sets intersect in an even number of elements.

As in the solutions of Problems 59 and 114, let us identify  $\mathcal{A}^* = \mathcal{A}_0 \cup \mathcal{A}_1^*$  with subset  $S$  of the vector space  $V = \mathbb{Z}_2^{n+\ell}$ , by letting each set  $A \in \mathcal{A}^*$  correspond to its characteristic vector. More precisely, let  $S_0$  be the set of vectors corresponding to the sets in  $\mathcal{A}_0$  and let  $S_1^*$  be the set of vectors corresponding to the sets in  $\mathcal{A}_1^*$ . Clearly,  $S = S_0 \cup S_1^*$  is a self-orthogonal set.

Let  $U$ ,  $U_0$  and  $U_1$  be spanned by  $S$ ,  $S_0$  and  $S_1^*$ , respectively, so that  $U = U_0 + U_1$ . Since  $A_i^*$  is the only member of  $\mathcal{A}_0 \cup \mathcal{A}_1^*$  that contains  $n+i$ ,  $S_1^*$  is an independent set and we have  $U_0 \cap U_1 = \{0\}$ , so  $U$  is the direct sum of  $U_0$  and  $U_1$ :  $U = U_0 \oplus U_1$ . Appealing to the independence of  $S_1^*$ , we see that  $\dim(U_1) = |S_1^*| = \ell$ . Also,  $U = U_0 \oplus U_1$  is a self-orthogonal subspace of  $V$  so, as shown in the solution of

Problem 114, its dimension is at most  $\lfloor (\dim(V))/2 \rfloor = \lfloor (n + \ell)/2 \rfloor$ :

$$\dim(U) = \dim(U_0) + \dim(U_1) = \dim(U_0) + \ell \leq \lfloor (n + \ell)/2 \rfloor,$$

i.e.,

$$\dim(U_0) \leq \lfloor (n - \ell)/2 \rfloor.$$

Finally,  $S_0$  consists of at most as many vectors as the subspace  $U_0$ , i.e.,  $k \leq 2^{\dim(U_0)}$ , implying inequality (50).

As we remarked earlier, it needs only a little work to show that for  $n \geq 6$  inequality (50) implies that  $k + \ell \leq 2^{n/2}$  if  $n$  is even, and  $k + \ell \leq 2^{(n-1)/2} + 1$  if  $n$  is odd.  $\square$

## 116. Even Clubs

*Suppose that in a town with population  $n$ , every club has an even number of members and any two clubs have an odd number of members in common. Then the maximum of the number of clubs is  $n$  if  $n$  is odd, and  $n - 1$  if  $n$  is even.*

*Proof.* For  $n$  odd, the assertion is equivalent to the one in Problem 59: simply replace each club by its complement. Then each club has an odd number of elements, and any two have an odd number of members in common. Thus, for  $n$  odd, the maximum is indeed  $n$ .

To show that for  $n$  even the maximum is  $n - 1$ , let  $\{A_1, A_2, \dots, A_\ell\}$  be even subsets of  $[n]$ , with any two intersecting in an odd number of elements. For  $1 \leq i \leq \ell$ , set  $A_i^* = A_i \cup \{n + 1\}$ , i.e., for each  $i$ , add the *same* element to  $A_i$  to obtain  $A_i^*$ . Then  $\{A_i^*\}_{i=1}^\ell$  is a system of odd subsets, with any two intersecting in an even number of elements. From this we can almost, but not quite, deduce that  $\ell \leq n - 1$ . However, we can save the day by adding two even sets to this system, preserving the condition that any two intersect in an even number of elements, namely the empty set and the set  $[n]$ . Therefore we have a collection of  $k = 2$  even subsets and  $\ell$  odd subsets of  $[n + 1]$  such that any two meet in an even number of elements. Appealing to inequality (50) we find that in the solution of Problem 115,  $k = 2 \leq 2^{\lfloor ((n+1)-\ell)/2 \rfloor}$ , implying  $n + 1 - \ell \geq 2$ , i.e.,  $\ell \leq n - 1$ .  $\square$

## 117. Covering the Sphere

If the  $n$ -dimensional unit sphere  $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : \|\mathbf{x}\| = 1\}$  is covered by  $n + 1$  sets, each of which is either open or closed, then one of the sets contains a pair of antipodal points, i.e., points  $x$  and  $-x$ .

*Proof.* This assertion is just a slight extension of the classical LSB theorem of Lusternik and Schnirelmann (1930) and Borsuk (1933) we presented in Problem 41, in which the  $n + 1$  sets are taken to be closed. This extension was noted by Greene (2003).

Let us see first that the case of open cover is an easy consequence of the original LSB theorem. Indeed, suppose that  $S^n = \bigcup_{i=1}^{n+1} U_i$ , with each  $U_i$  open. Let  $\lambda > 0$  be a Lebesgue number of this cover, i.e., such that for every point  $\mathbf{x} \in S^n$  there is an index  $i = i(\mathbf{x})$  such that the closed ball  $B_{\mathbf{x}} = \overline{B}(\mathbf{x}, \lambda)$  of centre  $\mathbf{x}$  and radius  $\lambda$  is contained in  $U_i$ . Since  $S^n$  is compact, the closed cover  $S^n = \bigcup_{\mathbf{x} \in S^n} B_{\mathbf{x}}$  contains a finite subcover. For  $1 \leq j \leq n + 1$ , let  $V_j$  be the union of the closed balls in this finite subcover that are contained in  $U_j$ . In particular,  $V_j$  is a closed subset of  $U_j$  and  $\bigcup_{j=1}^{n+1} V_j = S^n$ . Hence, by the LSB theorem, for some  $j$  the set  $V_j \subset U_j$  contains a pair of antipodal points, as claimed.

Let us turn to the general case. Let  $V_1, \dots, V_m$  be open subsets of  $S^n$  and  $U_{m+1}, \dots, U_{n+1}$  closed subsets with  $\bigcup_{i=1}^m V_i \cup \bigcup_{j=m+1}^{n+1} U_j = S^n$ . Suppose that none of these  $n + 1$  sets contains a pair of antipodal points. Since each  $V_i$  is compact, for some  $\varepsilon > 0$  each of them has diameter less than  $2 - \varepsilon$ . For  $1 \leq i \leq m$ , let  $U_i$  be the open subset of  $S^n$  consisting of the points at distance less than  $\varepsilon/2$  from  $V_i$ . Since  $S^n = \bigcup_{i=1}^{n+1} U_i$  is an open cover of  $S^n$ , as we have just seen, for some  $i$  the set  $U_i$  contains a pair of antipodal points. Since  $U_1, \dots, U_m$  do not contain antipodal pairs, this index  $i$  is greater than  $m$ , i.e., one of the original sets  $U_{m+1}, \dots, U_{n+1}$  contains a pair of antipodal points, completing our proof.  $\square$

K. Borsuk, Drei Sätze über die  $n$ -dimensionale euklidische Sphäre, *Fund. Math.* **20** (1933), 177–190.

J.E. Greene, A new short proof of Kneser's conjecture, *Amer. Math. Monthly* **109** (2002), 918–920.

L. Lusternik and L. Schnirelmann, *Topological Methods in Variational Calculus* (in Russian), Issledovatel'kii Institut Matematiki i Mehaniki pri O.M.G.U., Moscow, (1930).

## 118. The Kneser Graph: Lovász's Theorem

*Given natural numbers  $n$  and  $k$ , partition the  $n$ -subsets of  $\{1, 2, \dots, 2n + k\}$  into  $k + 1$  subsets  $A_1, \dots, A_{k+1}$ . Then one of the sets  $A_i$  contains two disjoint  $n$ -sets.*

*Proof.* This assertion was conjectured by Kneser in 1955, and was solved only twenty-three years later by Lovász by making use of some basic results of algebraic topology. Kneser's conjecture (by now, Lovász' theorem) has the following attractive reformulation in terms of the chromatic number of the so-called *Kneser graph*  $K(n, k)$ . This graph  $K(n, k)$  has  $\binom{2n+k}{n}$  vertices, namely all  $n$ -subsets of the set  $[2n+k] = \{1, 2, \dots, 2n+k\}$  (or any set of  $2n+k$  elements), and two vertices are joined by an edge if the corresponding  $n$ -subsets are disjoint. Note that for  $k$  fixed and  $n$  large this graph has a rather large independent set: all  $n$ -subsets containing a fixed element are independent, and there are  $\binom{2n+k-1}{n-1}$  such elements. Thus close to half, precisely  $n/(2n+k)$ , of the vertices form an independent set. Also,  $K(n, k)$  is certainly  $k+2$ -colourable: as colour classes we may take the collections of  $n$ -subsets with minimal element  $i$  for  $i = 1, \dots, k+1$  and the collection of  $n$  subsets contained in the  $2n-1$ -set  $\{k+2, k+3, \dots, 2n+k\}$ . What Kneser conjectured, and Lovász proved, was that this simple-minded colouring cannot be improved:  $K(n, k)$  has chromatic number  $k+2$ , although it contains many independent sets of almost half size.

Shortly after Lovász gave his beautiful proof of Kneser's conjecture, Bárány gave a remarkably short proof by combining the fundamental theorem of Lusternik, Schnirelmann and Borsuk with a theorem of Gale. Recently, to the great surprise of the many admirers of Bárány's short and beautiful proof, Greene has done what nobody thought could be done: he further simplified the proof by deducing Kneser's conjecture from the slight extension of the LSB theorem given in Problem 117. It is this gem of a proof that we are about to give.

Let  $A_1, \dots, A_{k+1}$  be a partition of the  $n$ -subsets of a set  $X$  of  $2n+k$  points placed in general position on the standard unit sphere  $S^{k+1}$ . For  $1 \leq i \leq k+1$ , let

$$U_i = \{\mathbf{x} \in S^{k+1} : H(\mathbf{x}) \text{ contains some } \sigma \in A_i\},$$

where  $H(\mathbf{x}) = \{\mathbf{y} \in S^{k+1} : \langle \mathbf{x}, \mathbf{y} \rangle > 0\}$  is the open half-sphere with centre at  $\mathbf{x}$ . Clearly, each  $U_i$  is an open subset of  $S^{k+1}$ . Set

$$V_0 = S^{k+1} \setminus \bigcup_{i=1}^{k+1} U_i,$$

so that  $V_0$  is closed and  $S^{k+1} = V_0 \cup \bigcup_{i=1}^{k+1} U_i$ .

By the extension of the theorem of Lusternik, Schnirelmann and Borsuk in Problem 117, one of the  $k+2$  sets of this cover contains a pair of antipodal points.

We claim that this set is not  $V_0$ . Indeed, a point  $\mathbf{x} \in S^{k+1}$  is in  $V_0$  if, and only if, the half-sphere  $H(\mathbf{x})$  contains at most  $n-1$  points of  $X$ . Hence, if we had  $\pm \mathbf{x} \in V_0$ , then at least  $2n + k - 2(n-1) = k+2$  points of  $X$  would be on the hyperplane orthogonal to  $\mathbf{x}$ , contradicting the fact that the points of  $X$  are in general position.

Consequently, there is an  $i$  such that the set  $U_i$  contains a pair  $\pm \mathbf{x}$  of antipodal points. This means that the open hemisphere  $H(\mathbf{x})$  contains an  $n$ -subset  $\sigma \in A_i$ , and the hemisphere  $H(-\mathbf{x})$  also contains an  $n$ -subset  $\tau \in A_i$ . But then  $\sigma$  and  $\tau$  are disjoint  $n$ -subsets belonging to the same set  $A_i$ , completing our proof.  $\square$

**Further Notes.** Much research has been done on Kneser-type problems. In particular, Alon, Frankl and Lovász, Kříž, Matoušek and Ziegler proved deep results for hypergraphs, and Matoušek showed that in many applications a particular case of Tucker's lemma can be used instead of the Lusternik–Schnirelmann–Borsuk theorem. The ‘de-topologization’ of the proofs was continued by Ziegler.

N. Alon, P. Frankl, and L. Lovász, The chromatic number of Kneser hypergraphs. *Trans. Amer. Math. Soc.*, **298** (1986), 359–370.

I. Bárány, A short proof of Kneser's conjecture, *J. Combin. Theory Ser. A* **25** (1978), 325–326.

D. Gale, Neighboring vertices on a convex polyhedron, in *Linear Inequalities and Related Systems*, H.W. Kuhn and A.W. Tucker (eds), Princeton University Press, (1956).

J.E. Greene, A new short proof of Kneser's conjecture, *Amer. Math. Monthly* **109** (2002), 918–920.

M. Kneser, Aufgabe 360, *Jber. Deutsch. Math.-Verein.*, 2. Abteilung **58** (1955), 27.

I. Kříž, Equivariant cohomology and lower bounds for chromatic numbers, *Trans. Amer. Math. Soc.* **333** (1992), 567–577.

L. Lovász, Kneser's conjecture, chromatic number, and homotopy, *J. Combin. Theory Ser. A* **25** (1978), 319–324.

J. Matoušek, On the chromatic number of Kneser hypergraphs, *Proc. Amer. Math. Soc.* **130** (2002), 2509–2514.

J. Matoušek, A combinatorial proof of Kneser's conjecture, *Combinatorica* **24** (2004), 163–170.

G.M. Ziegler, Generalized Kneser coloring theorems with combinatorial proofs, *Invent. Math.* **147** (2002), 671–691.

## 119. Partitions into Bricks

Let  $\mathbf{A} = A_1 \times \cdots \times A_n$  be the product of non-empty finite sets  $A_1, A_2, \dots, A_n$ . Then  $\mathbf{A}$  cannot be partitioned into fewer than  $2^n$  bricks, products of non-empty proper subsets of the factor spaces.

*Proof.* Let  $\mathbf{A} = \mathbf{B}_1 \cup \cdots \cup \mathbf{B}_m$  be a partition of  $\mathbf{A}$  into bricks, and let  $\mathbf{R}$  be a random odd product set obtained by picking one of the  $2^{|A_i|-1}$  odd subsets  $R_i$  of  $A_i$  and taking  $\mathbf{R} = R_1 \times \cdots \times R_n$ . For  $1 \leq i \leq m$ , let  $X_i$  be the 0-1 random variable defined by the parity of  $|\mathbf{B}_i \cap \mathbf{R}|$ :

$$X_i = \begin{cases} 0 & \text{if } |\mathbf{B}_i \cap \mathbf{R}| \text{ is even,} \\ 1 & \text{if } |\mathbf{B}_i \cap \mathbf{R}| \text{ is odd.} \end{cases}$$

Note that, for a brick  $\mathbf{B} = B_1 \times \cdots \times B_n$ , the intersection  $\mathbf{B} \cap \mathbf{R}$  has an odd number of elements if and only if each intersection  $B_i \cap R_i$  is odd. Since  $\emptyset \neq B_i \neq A_i$ , the probability that  $B_i \cap R_i$  is odd is  $1/2$ . Therefore, as the sets  $R_1, \dots, R_n$  are chosen independently,

$$\mathbb{P}(X_i = 1) = 2^{-n}$$

for every  $i$ . Consequently, with  $X = \sum_{i=1}^m X_i$  we have

$$\mathbb{E}(X) = \sum_{i=1}^m \mathbb{E}(X_i) = \sum_{i=1}^m \mathbb{P}(X_i = 1) = m2^{-n}.$$

On the other hand,  $\sum_{i=1}^m X_i \geq 1$  since the sum  $\sum_{i=1}^m |\mathbf{B}_i \cap \mathbf{R}| = |\mathbf{R}|$  is odd and so for each  $\mathbf{R}$  there is at least one odd summand  $|\mathbf{B}_i \cap \mathbf{R}|$ . Hence,

$$1 \leq \mathbb{E}(X) = m2^{-n},$$

implying that  $m \geq 2^n$ . □

**Notes.** The question above was posed by Kearnes and Kiss, and the remarkably beautiful proof above was found by Alon, Bohman, Holzman and Kleitman.

N. Alon, T. Bohman, R. Holzman and D.J. Kleitman, On partitions of discrete boxes, in *Kleitman and Combinatorics: a Celebration* (Cambridge, MA, 1999), *Discrete Math.* **257** (2002), 255–258.

K.A. Kearnes and E.W. Kiss, Finite algebras of finite complexity, *Discrete Math.* **207** (1999), 89–135.

## 120. Drawing Dense Graphs

*Every drawing of a graph with  $n$  vertices and  $m > 5n$  edges contains more than  $4m^3/243n^2 > m^3/(8n)^2$  pairs of crossing edges.*

*Proof.* Our task is to show that if a graph  $G$  has  $n$  vertices and  $m > 5n$  edges then  $\text{cr}(G) > 4m^3/243n^2$ , where  $\text{cr}(G)$ , the *crossing number* of  $G$ , is the minimal number of pairs of crossing edges of  $G$  in its drawing in the plane.

Since a planar graph of order  $n$  has at most  $3n - 6$  edges, a graph with  $n$  vertices and  $m$  edges has crossing number at least  $m - 3n + 6 > m - 3n$ . This is the only inequality we shall need to prove the result along the lines of attack indicated in the *Hint*.

Thus, let  $G$  be a graph of order  $n$  with  $m > 5n$  edges, and fix a drawing of  $G$  in the plane with  $\text{cr}(G)$  crossings (pairs of crossing edges). Set  $p = 9n/2m$ , and let  $G_p$  be the random subgraph of  $G$  obtained by selecting the *vertices* independently, with probability  $p$ . Let  $X(G_p)$  be the number of crossings  $G_p$  has in our drawing of  $G$ . Then  $X(G_p) \geq \text{cr}(G_p)$ ; consequently,

$$X(G_p) > e(G_p) - 3|G_p|.$$

To complete our proof we just take expectations in this inequality. Since a crossing in our drawing of  $G$  is counted in  $X(G_p)$  only if all four vertices of the two crossing edges are selected, it follows that  $\mathbb{E}(X(G_p)) = p^4 \text{cr}(G)$ . Also,  $\mathbb{E}(e(G_p)) = p^2 m$  and  $\mathbb{E}(|G_p|) = pn$ . Hence, the inequality above implies

$$p^4 \text{cr}(G) > p^2 m - 3pn,$$

and so

$$\text{cr}(G) > \frac{m}{p^2} - \frac{3n}{p^3} = \frac{m^3}{n} \left( \left( \frac{2}{9} \right)^2 - 3 \left( \frac{2}{9} \right)^3 \right) = \frac{4}{243} \frac{m^3}{n^2},$$

as required. □

**Notes.** This beautiful result was proved by Leighton, and Ajtai, Chvátal, Newborn and Szemerédi, but the elegant and simple proof above is from a paper of Pach and Tóth. Nevertheless, it is thought that the same proof was found by Füredi, Lovász and Matoušek (and perhaps others) as well, inspired by Székely's breakthrough paper concerning an applications of this result (see Problems 121, 122).

Among many other results, Pach and Tóth showed also that the crossing number of a graph with  $n$  vertices and  $m$  edges is greater than  $0.029m^3/n^2 - 0.9n$ , and made use of this to prove that the number of incidences between  $n$  points and  $m$



lines is at most  $2.57m^{2/3}n^{2/3} + m + n$ , improving an earlier result of Szemerédi and Trotter.

M. Ajtai, V. Chvátal, M. Newborn and E. Szemerédi, Crossing-free subgraphs, *Ann. Discrete Math.* **12** (1982), 9–12.

F.T. Leighton, *Complexity Issues in VLSI: optimal layouts for the shuffle-exchange graph and other networks*, MIT Press, (1983).

J. Pach and G. Tóth, Graphs drawn with few crossings per edge, *Combinatorica* **17** (1997), 427–439.

L. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Combin. Prob. and Comput.* **6** (1997), 353–358.

E. Szemerédi and W.T. Trotter, Extremal problems in discrete geometry, *Combinatorica* **3** (1983), 381–392.

## 121. Unit Distances: Székely's Theorem

*Given  $n$  points in the plane they determine at most  $4n^{4/3}$  unit distances.*

*Proof.* Let us apply induction on  $n$ . For  $n \leq 8$  there is nothing to prove, so let us turn to the induction step. Given  $n > 8$  points in the plane with  $m$  unit distances, if one of these points is at distance 1 from no more than 10 points, then  $m \leq 10 + 4(n-1)^{4/3} < 4n^{4/3}$ , as required. Hence we may assume that each of our  $n$  points is at distance 1 from at least 11 other points. (In fact, instead of 11, three would do in what follows.)

Let us draw a multigraph in the plane with this set of vertices as follows. Draw a unit circle about every point: such a circle contains at least three (in fact, at least 11) other points. Join the consecutive points on these unit circles by the arcs of these circles. In this way we obtain a multigraph with  $2m$  edges, in which some pairs of points are joined by two edges, but never by three. Discarding one of each pair of double edges, we obtain a graph drawn in the plane with at least  $m$  edges. The number of crossings of this graph is at most  $n(n-1)$  since any two circles intersect in at most two points. Therefore either  $m \leq 5n < 4n^{4/3}$ , or else, by the result in Problem 120,  $m^3/64n^2 < n^2$ , and so  $m < 4n^{4/3}$ .  $\square$

**Notes.** In 1946, Erdős conjectured that the number of unit distances among  $n$  points in the plane is at most  $n^{1+o(1)}$  and proved that this number is at most  $cn^{3/2}$ . After various improvements by Józsa and Szemerédi, and Beck and Spencer, the best bound to date,  $cn^{4/3}$ , was proved by Spencer, Szemerédi and Trotter. Recently, Székely found the exceptionally beautiful and simple proof we have just presented. In fact, the proof is reproduced from a review article about Erdős, in which it was conjectured that this was the last proof Erdős judged to have come straight from The Book. In fact, there is no need to apply induction: the main part of the argument implies that there are at most  $4n^{4/3} + n$  unit distances. We have eliminated  $n$  for cosmetic reasons only.

J. Beck and J. Spencer, Unit distances, *J. Combin. Theory Ser. A* **37** (1984), 231–238.

B. Bollobás, To Prove and Conjecture: Paul Erdős and his Mathematics, *Amer. Math. Monthly* **105** (1998), 209–237.

P. Erdős, On sets of distances of  $n$  points, *Amer. Math. Monthly* **53** (1946), 248–250.

S. Józsa and E. Szemerédi, The number of unit distances on the plane, in *Infinite and Finite Sets* Coll. Math. Soc. J. Bolyai **10** (1973), 939–950.

J. Spencer, E. Szemerédi and W.T. Trotter, Unit distances in the Euclidean plane, in *Graph Theory and Combinatorics*, B. Bollobás (ed.), Academic Press, (1984), pp. 293–308.

L. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Combin. Prob. and Comput.* **6** (1997), 353–358.



## 122. Point–Line Incidences

*There are at most  $2(2n\ell)^{2/3} + \ell$  incidences among  $n$  points and  $\ell$  lines in the plane.*

*Proof.* Let  $N$  be a set of  $n$  points and  $L$  a set of  $\ell$  lines in the plane, with  $m$  incidences among them. Let  $L_2$  be the set of lines incident with at least two points, and set  $\ell_2 = |L_2|$  and  $\ell_1 = \ell - \ell_2$ . Let  $G$  be the graph with vertex set  $N$  (the set of ‘nodes’) in which the edge set is the set of straight-line segments joining two points, provided they do not contain other points. (Thus the edge set is the set of segments determined by neighbouring points on the lines.) Then  $G$  has  $n$  vertices and at least  $m - \ell$  edges, since a line in  $L$  incident with  $i \geq 1$  points gives  $i - 1$  edges. Also, the number of crossings of pairs of edges of  $G$  is at most  $\binom{\ell_2}{2}$ . Hence, by the theorem of Leighton, Ajtai, Chvátal, Newborn and Szemerédi in Problem 120, either  $m - \ell \leq 5n$  or else

$$(m - \ell_2)^3 / 8n^2 < \binom{\ell_2}{2},$$

implying that

$$m < (32n^2\ell_2^2)^{1/3} + \ell_2 < 2(2n\ell)^{2/3} + \ell.$$

□

### 123. Geometric Graphs without Parallel Edges

*The maximal number of edges in a geometric graph of order  $n$  without parallel edges is  $2n - 2$ .*

*If  $n \geq 4$  is even then there is a geometric graph with  $n$  vertices,  $2n - 2$  edges and minimal degree 3 that does not contain two parallel edges.*

*Proof.* Let  $G = (V, E)$  be a geometric graph with  $n$  vertices and  $m$  edges, no two of which are parallel. We may assume that the minimal degree of  $G$  is at least 3, since otherwise we may replace  $G$  by a graph with fewer vertices. (This assumption is far from vital, but as it comes free, we shall make it.) Orient the edges of  $G$  in any way; let  $\vec{E}$  be the set of oriented edges. Consider a unit circle  $S$ , and for each edge  $\vec{e} = \overrightarrow{ab} \in \vec{E}$  mark two antipodal points  $a$  and  $b$  of  $S$  such that the oriented diameter  $\overrightarrow{ab}$  is parallel to the oriented edge  $\vec{e}$  of  $G$  (and, needless to say, has the same orientation as  $\vec{e}$ ).

**Claim 1.** *The pattern  $abab$  does not occur as a circular subsequence among the labels on  $S$ .*

Rather than writing out a few cases in order to show this, we leave the proof of this claim to the reader.

In view of our construction and Claim 1, in order to solve our main question, it suffices to prove the following claim.

**Claim 2.** *Let  $C$  be a cyclic word of length  $2m$  with symbols  $1, 2, \dots, n$  such that no two antipodal symbols are the same, and the pattern  $abab$  is not a subsequence of  $C$ . Then  $m \leq 2n - 2$ .*

To prove Claim 2, let  $E$  be the equator separating the upper and lower hemispheres of a unit sphere. Let  $P$  be a regular  $2m$ -gon inscribed into  $E$ , and identify  $C$  with the vertices of  $P$  in a way that preserves the cyclic order. For each symbol  $i$ , let  $U_i$  be the convex spherical polygon in the upper hemisphere whose vertices are the vertices of  $P$  labelled  $i$ , and let  $V_i$  be the reflection of  $U_i$  in the centre of our unit sphere (see Figure 81).

As each symbol occurs at least three times, the  $U_i, V_j$  are proper, non-degenerate polygons. (This is again unimportant, but we do have it since the minimal degree of  $G$  is at least 3.) Since  $C$  does not contain the subpattern  $abab$ , for  $i \neq j$  the closed polygons  $U_i$  and  $U_j$  are disjoint, and so are  $V_i$  and  $V_j$ . Also, since  $G$  contains at most one edge joining  $i$  to  $j$ , two polygons  $U_i$  and  $V_j$  are either disjoint or share a single vertex.

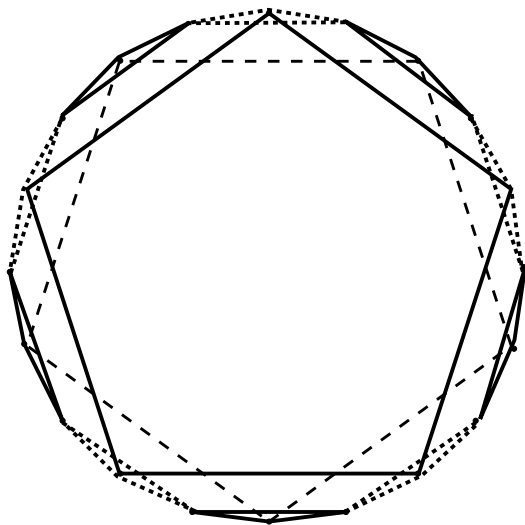


Fig. 81. A schematic straight-line drawing of an arrangement of spherical polygons  $U_i$  and  $V_j$ ,  $i, j = 1, \dots, 6$ : the sides of  $U_i$  are drawn with solid lines, those of  $V_i$  are dotted.

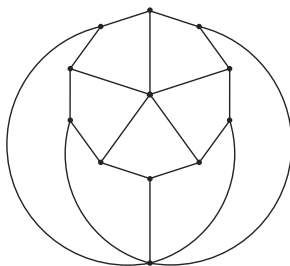
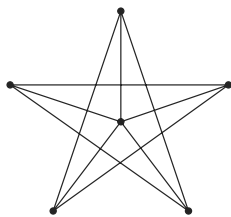


Fig. 82. The bipartite graph  $H$  corresponding to the arrangement in Figure 81.

After this preparation, let us turn to the proof of Claim 2. Let  $H$  be the bipartite graph with vertex sets  $\{U_i : 1 \leq i \leq n\}$  and  $\{V_j : 1 \leq j \leq n\}$  in which  $U_i$  and  $V_j$  are joined by an edge if they share a vertex (see Figure 82).

Putting it another way,  $H$  is the ‘kissing graph’ of the convex spherical sets  $U_i, V_j$ ; in particular,  $H$  is planar. Now, as  $H$  is a bipartite planar graph with  $2n$  vertices, it has at most  $4n - 4$  edges. On the other hand, two edges of  $H$  correspond to every edge  $ij$  of  $G$ :  $U_i V_j$  and  $U_j V_i$ . Hence,  $2m \leq 4n - 4$ , proving the claim, and so completing the solution of the main part of our problem.

Fig. 83. The geometric graph  $S_6$ .

For  $n \geq 4$  even, let  $S_n$  be the geometric graph obtained from a regular star polygon with  $n - 1$  vertices by adding the centre and joining that to each of the other  $n - 1$  vertices, as in Figure 83. This graph has  $2n - 2$  edges, and does not contain two parallel edges.  $\square$

**Notes.** The result in the question was conjectured by Yaakov Kupitz, and proved by Katchalski and Last, and Valtr. Recently, Pinchasi found two beautiful and simple proofs of this result, one of which was presented above. For numerous attractive results about geometric graphs, see the monograph of Pach and Agarwal.

M. Katchalski and H. Last, On geometric graphs with no two edges in convex position, *Discrete Comput. Geom.* **19** (1998), 399–404.

Y.S. Kupitz, On pairs of disjoint segments in convex position in the plane, in *Convexity and Graph Theory* (Jerusalem, 1981), North-Holland Math. Stud., **87**, North-Holland, (1984), pp. 203–208

J. Pach and P.K. Agarwal, *Combinatorial Geometry*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley, (1995), xiv + 354 pp.

R. Pinchasi, A note on geometric graphs with no pair of parallel edges, manuscript

R. Pinchasi, A note on a circular Davenport–Schinzel sequence, manuscript

P. Valtr, On geometric graphs with no  $k$  pairwise parallel edges, *Discrete Comput. Geom.* **19** (1998), 461–469.

## 124. Shortest Tours

The square length  $s(\mathcal{P})$  of a polygonal path  $\mathcal{P}$  is the sum of the squares of the side-lengths of  $\mathcal{P}$ . Let  $W$  be a set of  $n \geq 3$  points in a unit square. Then there is a closed polygonal path  $\mathcal{P}$  with vertex set  $W$  such that  $s(\mathcal{P}) \leq 4$ . If 4 is replaced by any smaller number, this assertion is no longer true.

*Proof.* We start by proving the following Claim.

**Claim.** Let  $T = ABC$  be a right-angled isosceles triangle, with hypotenuse  $AB$ . Given a finite set  $W$  of points in  $T$ , there is a polygonal path  $\mathcal{P}$  from  $A$  to  $B$  with internal vertex set  $W$  such that  $s(\mathcal{P}) \leq |AB|^2$ .

*Proof.* Let us apply induction on  $n = |W|$ . If  $W$  consists of a single point  $P$  then we have a unique polygonal path  $\mathcal{P} = APB$ , with the angle at  $P$  at least  $\pi/2$ , so

$$s(\mathcal{P}) = |AP|^2 + |BP|^2 \leq |AB|^2.$$

To prove the induction step, let  $n \geq 2$  and suppose that the assertion holds for up to  $n - 1$  points. Let  $D$  be the foot of the height of the triangle  $T$  from  $C$ . The height  $CD$  cuts  $T$  into two congruent triangles,  $CDA$  and  $CDB$ . Partition  $W$  into sets  $W_A$  and  $W_B$ , with  $W_A$  in  $CDA$  and  $W_B$  in  $CDB$ .

First, suppose that  $W_A$  and  $W_B$  can be chosen to be non-empty. Then, by the induction hypothesis, there are polygonal paths  $\mathcal{P}_A$  and  $\mathcal{P}_B$ , the first through  $W_A$  from  $A$  to  $C$ , and the second through  $W_B$  from  $C$  to  $B$ . As in Figure 84, let  $\mathcal{P}$  be the polygonal path obtained from the concatenation of  $\mathcal{P}_A$  and  $\mathcal{P}_B$  by replacing the two-step path  $P_\ell C P_f$  in the middle by the single step  $P_\ell P_f$ , where  $P_\ell$  is the last point of  $W$  on  $\mathcal{P}_A$  and  $P_f$  is the first point of  $W$  on  $\mathcal{P}_B$ . Since the angle at  $C$  is at most  $\pi/2$ ,

$$|P_\ell C|^2 + |C P_f|^2 \geq |P_\ell P_f|^2,$$

so

$$s(\mathcal{P}) = s(\mathcal{P}_A) + s(\mathcal{P}_B) + |P_\ell P_f|^2 - |P_\ell C|^2 - |C P_f|^2 \leq s(\mathcal{P}_A) + s(\mathcal{P}_B) \leq |AB|^2,$$

as claimed.

Morally, this completes the proof, since all we have to do is to get rid of the obviously harmless assumption that  $W_A$  and  $W_B$  are non-empty. This can be done in many ways. For example, we may go on halving the triangles until we do separate  $W$  into two non-empty sets, and give the argument for that triangle.



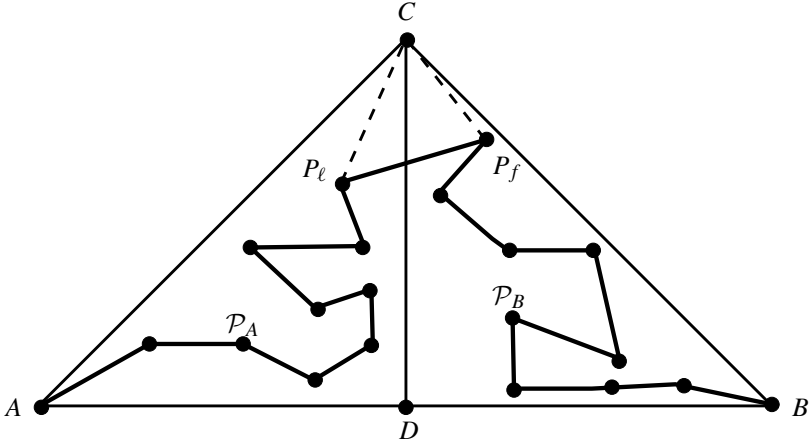
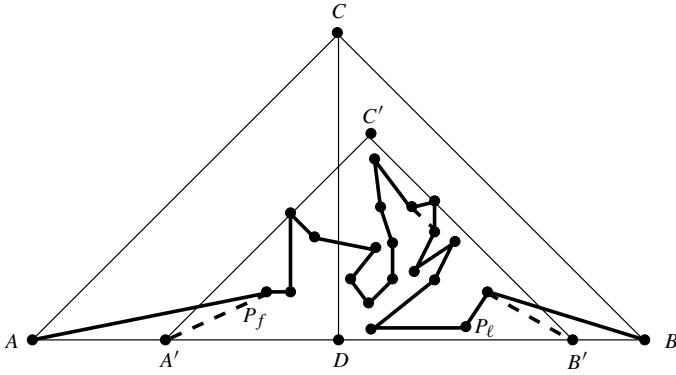


Fig. 84. A route found by the algorithm.

Fig. 85. The paths  $\mathcal{P}' = A'P_f \dots P_lB'$  and  $\mathcal{P} = AP_f \dots P_lB$ .

Another way is to show that it suffices to prove the result in the case when both  $AC$  and  $BC$  contain points of  $W$ ; then  $W_A$  and  $W_B$  can be chosen to be non-empty, and we are done again. Let then  $A'B'C'$  be the minimal triangle containing  $W$ , with  $A'B'$  on  $AB$  and the triangle  $T' = A'B'C'$  similar to  $T = ABC$ . Let  $\mathcal{P}'$  be a polygonal path from  $A'$  to  $B'$  through  $W$  with  $s(\mathcal{P}') \leq |A'B'|^2$ , say,  $\mathcal{P}' = A'P_f \dots P_lB'$ . Then, with  $\mathcal{P} = AP_f \dots P_lB$ , as in Figure 85, we have

$$s(\mathcal{P}) \leq |A'B'|^2 + |AP_f|^2 - |A'P_f|^2 + |P_lB|^2 - |P_lB'|^2 \leq |AB|^2,$$

completing the proof of the Claim.  $\square$

From the Claim, it is but a short step to the assertion about the unit square. Indeed, cut the unit square into two congruent triangles, take the two polygonal paths guaranteed by the Claim, and shorten them appropriately, as in Figure 86.

□

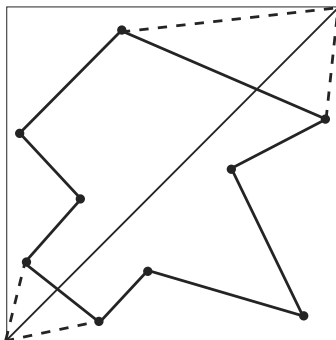


Fig. 86. Uniting two polygonal paths.

## 125. Density of Integers

*The density of integers divisible by precisely one member of a set  $A$  of positive integers not containing 1 is at most  $1/2$ . Furthermore, this density is  $1/2$  if and only if  $A = \{2\}$  or  $A = \{2, a\}$ , with  $a \geq 3$  odd.*

*Proof.* Let  $A = \{a_1, \dots, a_k\}$  be a finite set of integers greater than 1, and set  $M_A = \{m \in \mathbb{N} : m \text{ is the multiple of exactly one } a_i\}$ . The density of  $M_A$  is  $\delta(M_A) = \lim_{n \rightarrow \infty} |M_A \cap [n]|/n$ . Our task is to show that  $\delta(M_A) \leq 1/2$ . Note first that we may assume that no  $a_i$  divides another.

In spite of the appearance of a limit above, the assertion concerns only finite sets. Indeed, for sets  $X, Y$ , with  $Y$  finite, let  $\delta(X|Y) = |X \cap Y|/|Y|$  be the density of  $X$  in  $Y$ . Then, for the least common multiple  $N$  (just as for every common multiple) of  $a_1, \dots, a_k$ , we have  $\delta(M_A) = \delta(M_A|[N])$ .

Let  $A_i$  be the set of multiples of  $a_i$  between 2 and  $N$ . If the  $a_i$  are pairwise relatively prime then  $A_1, \dots, A_k$  are independent subsets of  $[N]$  with respect to the density  $\delta(\cdot|[N])$ , so

$$\delta(M_A) = \sum_{i=1}^k \frac{1}{a_i} \prod_{j \neq i} (1 - 1/a_j).$$

With  $R = \prod_{i=1}^k \left( (1 - \frac{1}{a_i}) + \frac{1}{a_i} \right) = 1$  and  $S = \prod_{i=1}^k \left( (1 - \frac{1}{a_i}) - \frac{1}{a_i} \right) \geq 0$ , we thus find that

$$\delta(M_A) \leq (R - S)/2 \leq 1/2,$$

with equality if and only if  $a_1 = 2$  and  $k \leq 2$ , as claimed.

Now we are coming to the heart of the proof: we have to show that if some two of the  $a_i$  are not relatively prime then  $\delta(M_A) < 1/2$ . We shall prove this by induction on  $\|A\| = \sum_{i=1}^k a_i$ . For  $\|A\| \leq 2$  there is nothing to prove, so we pass on to the induction step. Let  $p$  be a prime such that  $A = \{pb_1, \dots, pb_\ell, c_{\ell+1}, \dots, c_k\}$ , with  $\ell \geq 2$  and no  $c_i$  divisible by  $p$ . Set  $B = \{b_1, \dots, b_\ell\}$  and  $C = \{c_{\ell+1}, \dots, c_k\}$ .

If  $C = \emptyset$  then we are done by the induction hypothesis. Otherwise, let  $[N] = P \cup Q$ , with every element of  $P$  and no element of  $Q$  divisible by  $p$ . Then  $\delta(M_A|P) \leq \delta(M_{B \cup^* C}) \leq 1/2$ , where  $B \cup^* C$  denotes the multiset (or sequence) obtained from  $B$  and  $C$ , with possible repetitions, and  $\delta(M_A|Q) = \delta(M_C|Q) = \delta(M_C) \leq 1/2$ . It is easily seen that we cannot have equality in both cases, so  $\delta(M_A) = \delta(M_A|[N]) < 1/2$ . □

**Notes.** This assertion was proved by Lubell, when he solved a problem he had heard from Erdős.

D. Lubell, On a density problem of Erdős, *Acta Arith.* **19** (1971), 306–309.

## 126. Black and White Sheep: Kirchberger's Theorem

Let  $A$  and  $B$  be finite sets in  $\mathbb{R}^n$ , with  $|A| + |B| \geq n + 2$ . We say that a hyperplane  $H$  in  $\mathbb{R}^n$  strictly separates  $A$  from  $B$  if  $A$  is in one of the open half-spaces determined by  $H$  and  $B$  is in the other. Then there is a hyperplane  $H$  in  $\mathbb{R}^n$  that strictly separates  $A$  from  $B$  provided there is such a hyperplane for  $A'$  and  $B'$ , whenever  $A' \subset A$ ,  $B' \subset B$ , and  $|A'| + |B'| \leq n + 2$ .

*Proof.* Let us start with a simple observation concerning strict separation. Let  $V$  and  $W$  be disjoint compact convex sets in  $\mathbb{R}^n$ . Then there is a hyperplane  $H$  that strictly separates  $V$  from  $W$ . Indeed, as the sets are compact, their distance  $d(V, W) = \inf\{d(v, w) : v \in V, w \in W\}$  is attained at a pair of points: there are  $v_0 \in V$  and  $w_0 \in W$  such that  $d(v_0, w_0) = d(V, W)$ . But then the perpendicular bisector  $H$  of the segment  $[v_0, w_0]$  strictly separates  $V$  from  $W$ .

Since  $\text{conv}A$  and  $\text{conv}B$  are compact convex sets, in view of this observation all we have to prove is that if  $\text{conv}A \cap \text{conv}B \neq \emptyset$  then there are sets  $A' \subset A$  and  $B' \subset B$  with  $|A'| + |B'| \leq n + 2$  such that  $\text{conv}A' \cap \text{conv}B' \neq \emptyset$ . This assertion is clearly an extension of Carathéodory's theorem in Problem 28.

There are numerous ways of proving this: here we shall choose one that actually makes use of Carathéodory's theorem.

Suppose then that  $|A'| + |B'| = r + s$  is minimal, subject to the conditions  $A' \subset A$ ,  $B' \subset B$  and  $\text{conv}A' \cap \text{conv}B' \neq \emptyset$ . Suppose for a contradiction that  $r + s \geq n + 3$ . By Carathéodory's theorem, the convex hulls  $R = \text{conv}A'$  and  $S = \text{conv}B'$  are simplices. By translating the points, if necessary, we may assume that  $0 \in R \cap S$  (and so 0 is in the interior of these simplices). Let  $U = \text{lin } A'$  be the linear span of  $A'$  (and of  $R$ ); similarly, let  $V = \text{lin } B' = \text{lin } S$ . Then  $\dim(U) + \dim(V) = (r - 1) + (s - 1) \geq n + 1$ , so  $U \cap V$  contains the line  $\{tc : t \in \mathbb{R}\}$  through a point  $c \in R \cap S$ ,  $c \neq 0$ .

Let the half-line  $\{tc : t > 0\}$  meet the boundary of  $R$  in  $t_R c$ , and the boundary of  $S$  in  $t_S c$ . Assuming, as we may, that  $t_R \leq t_S$ , we find that the point  $t_R c \in R \cap S$  is on a face of  $R$ , so  $t_R c \in \text{conv } A'' \cap \text{conv } B'$ , where  $A'' \subset A'$ ,  $|A''| < |A'| = r$ . This contradicts the minimality of  $|A'| + |B'|$ , and so our proof is complete.  $\square$

**Notes.** Our formulation of Kirchberger's theorem is essentially that of Valentine.

P. Kirchberger, Über Tschebyscheffsche Annäherungsmethoden, *Mathematische Annalen* **57** (1903), 509–540.

F.A. Valentine, *Convex Sets*, McGraw Hill, (1964), ix + 238pp.

## 127. Chords of Convex Bodies

Let  $C \subset \mathbb{R}^n$  be a compact convex set with non-empty interior. Then there is a point  $c \in C$  such that

$$\|c - u\|/\|v - u\| \leq n/(n + 1),$$

whenever  $[u, v]$  of  $C$  is a chord through  $c$ .

*Proof.* For  $x \in C$ , let  $C_x$  be the dilation of  $C$  from  $x$  with ratio  $n/(n + 1)$ :

$$C_x = x + \frac{n}{n + 1}(C - x).$$

We claim that the intersection of all these sets  $C_x$  is non-empty. As noted in the solution of Problem 29, a trivial consequence of Helly's theorem for finite systems is that if an infinite family of compact convex sets in  $\mathbb{R}^n$  is such that any  $n + 1$  members of the family intersect, then they all do. Hence to prove our claim it suffices to check that any  $n + 1$  sets of the form  $C_x$ ,  $x \in C$ , have a non-empty intersection. Now, this is immediate since if  $x_0, \dots, x_n$  are points of  $C$  then  $\bar{x} = (x_0 + \dots + x_n)/(n + 1)$  is in the intersection  $\bigcap_{i=0}^n C_{x_i}$ :

$$\bar{x} = x_i + \frac{n}{n + 1} \left( \frac{1}{n} \sum_{j \neq i} x_j - x_i \right) \in x_i + \frac{n}{n + 1} (C - x_i)$$

for every  $i$ ,  $i = 0, \dots, n$ .

To complete the proof, let us show that every point  $c$  of the intersection  $\bigcap_x C_x$  will do for our 'central' point. Let  $[u, v]$  be a chord of  $C$  through  $c$ . Then since  $x \in C_u$ ,

$$c = u + \frac{n}{n + 1}(a - u)$$

for some  $a \in C$ . We claim that this  $a$  is a point of the chord  $[u, v]$ . This is certainly the case if  $a$  is  $u$ ,  $v$  or  $c$ . Otherwise, as  $a = \frac{n+1}{n}c - \frac{1}{n}u$ , the point  $a$  is on the line  $\ell$  of the segment  $[c, u]$ ; consequently  $a$  is a point of the intersection  $\ell \cap C$ , which is exactly  $[u, v]$ . Since  $a$  is point of the chord  $[u, v]$ , we have  $a = \lambda u + (1 - \lambda)v$  for some  $\lambda$  with  $0 \leq \lambda \leq 1$ . Substituting this into the expression for  $c$ , we find that

$$c = u + \frac{n}{n + 1}(\lambda u + (1 - \lambda)v - u) = u + \frac{n(1 - \lambda)}{n + 1}(v - u),$$

implying that

$$\frac{\|c - u\|}{\|v - u\|} = \frac{n(1 - \lambda)}{n + 1} \leq \frac{n}{n + 1}.$$

□

**Notes.** The result and the proof above are from L. Danzer, B. Grünbaum and V. Klee, Helly's theorem and its relatives, in *Convexity*, Proc. Symposia in Pure Math., vol. VII, American Math. Soc., (1963), pp. 101–180.

## 128. Neighbourly Polyhedra

*For every  $n$ , there are  $n$  internally disjoint convex polyhedra in  $\mathbb{R}^3$  such that any two of them share a face.*

*Proof.* We shall prove more: we shall construct a partition of the entire space  $\mathbb{R}^3$  into infinitely many disjoint convex sets, namely the Voronoi cells of a sequence of points, such that any two of the convex cells share a face.

Let  $P_1, P_2, \dots$  be points on the moment curve  $(x, x^2, x^3)$ . To make our calculations easier, we take a ‘sparse’ sequence:  $P_i = (n_i, n_i^2, n_i^3)$ , with  $n_1 \geq 100$  and  $n_{i+1} \geq n_i^2$  for  $i \geq 1$ . For example, we may take  $n_i = 10^{2^i}$ ,  $i = 1, 2, \dots$ . Let  $C_i$  be the Voronoi cell of  $P_i$ :

$$C_i = \{P \in \mathbb{R}^3 : |P - P_i| \leq |P - P_j| \text{ for all } j \neq i\}.$$

Our aim is to show that any two of these Voronoi cells share a face.

In order to reduce the clutter, for  $\alpha, \beta, \gamma = 1, 2, \dots$ , we write  $a = n_\alpha, b = n_\beta$  and  $c = n_\gamma$ . We shall assume that  $\alpha, \beta$  and  $\gamma$  are distinct, although we shall frequently emphasize this. For  $\alpha \neq \beta$ , the set of points equidistant from  $P_\alpha = (a, a^2, a^3)$  and  $P_\beta = (b, b^2, b^3)$  is the plane formed by the points  $(x, y, z)$  satisfying

$$(a - b)x + (a^2 - b^2)y + (a^3 - b^3)z = K_b, \quad (51)$$

where  $K_s = \frac{1}{2}(a^2 - s^2 + a^4 - s^4 + a^6 - s^6)$  for any  $s$ , with  $a$  being fixed. Therefore, to prove our assertion, we have to show that there is some point  $(x, y, z)$  satisfying (51) such that

$$(x - c)^2 + (y - c^2)^2 + (z - c^3)^2 > (x - a)^2 + (y - a^2)^2 + (z - a^3)^2 \quad (52)$$

whenever  $c \neq a, b$ . Indeed, if (52) holds then, as  $|P_\gamma| \rightarrow \infty$  as  $\gamma \rightarrow \infty$ , the difference between the two sides of (52) is greater than  $\varepsilon$  for some  $\varepsilon > 0$ , and so  $(x, y, z)$  is an interior point of a common face of  $C_\alpha$  and  $C_\beta$ .

Using (51) to eliminate  $z$  from (52) and then collecting terms, we find that our task is to find  $x$  and  $y$  satisfying

$$\begin{aligned} & \left( (a - c) - \left( \frac{a^3 - c^3}{a^3 - b^3} \right) (a - b) \right) x + \left( (a^2 - c^2) - \left( \frac{a^3 - c^3}{a^3 - b^3} \right) (a^2 - b^2) \right) y \\ & > K_c - K_b \left( \frac{a^3 - c^3}{a^3 - b^3} \right), \end{aligned}$$



which can be rewritten as

$$\frac{(a-c)(b-c)}{a^2+ab+b^2} ((a+b+c)x + (ab+c(a+b))y + p_{ab}(c)/2) > 0, \quad (53)$$

where

$$\begin{aligned} p_{ab}(c) = & (a^2 + ab + b^2)c^4 + (a^3 + 2a^2b + 2ab^2 + b^3)c^3 \\ & + (a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4 + a^2 + ab + b^2)c^2 \\ & + (a+b) \left( ab(a^2 + ab + b^2) + ab - 1 \right) c \\ & + ab \left( ab(a^2 + ab + b^2)ab - 1 \right). \end{aligned}$$

Now, (53) holds if  $S(a, b, c, x, y) = (a+b+c)x + (ab+c(a+b))y + p_{ab}(c)/2$  is positive whenever  $c < a < b$  or  $a < b < c$ , and negative whenever  $a < c < b$ .

Note that, trivially,  $p_{ab}(c) \geq a^2b^4$ , and, rather crudely, if  $a < c < b$  then  $p_{ab}(c) \leq 2c^2b^4$  since  $a < c < b$  implies that  $100 \leq a < a^2 \leq c < c^2 \leq b$ .

We claim that the pair  $(x, y) = (3ab^4, -\frac{3}{2}b^4)$  satisfies (53). Indeed, if  $c < a < b$  then

$$S(a, b, c, x, y) \geq 3ab^5 - (ab + c(a+b))\frac{3}{2}b^4 \geq 3ab^5 - 2ab^5 > 0,$$

if  $a < c < b$  then

$$S(a, b, c, x, y) \leq 4ab^5 - \frac{3}{2}cb^5 + 2c^2b^4,$$

and if  $a < b < c$  then

$$S(a, b, c, x, y) \geq 3acb^4 - 2cb^5 + a^2c^4 > 0.$$

Hence (53) does hold, and so any two of the Voronoi cells share a face.

In order to construct  $n$  internally disjoint convex polyhedra, any two of which share a face, let  $P_1, \dots, P_n$  be the first  $n$  (or any  $n$ ) of our points, and, for  $1 \leq i < j \leq n$ , let  $Q_{i,j}$  be a point in the interior of the common face of the Voronoi cells  $C_i$  and  $C_j$  belonging to  $P_i$  and  $P_j$ . Let  $C$  be a cube containing all the points  $P_i$ ,  $1 \leq i \leq n$ , and  $Q_{i,j}$ ,  $1 \leq i < j \leq n$ , in its interior, and set  $K_i = C_i \cap C$ ,  $1 \leq i \leq n$ . Then  $K_1, K_2, \dots, K_n$  are internally disjoint polyhedra and any two of them share a face.  $\square$

**Notes.** This well known problem is usually called *Crum's problem*, who posed it in 1947. The problem was solved by Besicovitch, whose solution was promptly generalized by Rado. The fame of the problem owes much to Littlewood, who popularized it in his *Miscellany*. The solution above is based on Dewdney and Vranich.

In fact, the problem and its solution go back to over forty years before Crum: Tietze answered a question of Stächel by constructing an infinite sequence of internally disjoint convex bodies in  $\mathbb{R}^3$ , any two of which have a common boundary of positive area. For an account of the many contributions to this problem, see pages 139–141 of Croft, Falconer and Guy.

A.S. Besicovitch, On Crum's problem, *J. London Math. Soc.* **22** (1947), 285–287.

H.T. Croft, K.F. Falconer and R.K. Guy, *Unsolved Problems in Geometry*, corrected reprint of the 1991 original, Problem Books in Mathematics, Unsolved Problems in Intuitive Mathematics, II. Springer-Verlag, (1994), xvi + 198 pp.

A.K. Dewdney and J.K. Vranich, A convex partition of  $\mathbb{R}^3$  with applications to Crum's problem and Knuth's post-office problem, *Utilitas Math* **12** (1977), 193–199.

J.E. Littlewood, *A Mathematician's Miscellany*, Methuen, (1953); see also its second, greatly enlarged edition: *Littlewood's Miscellany*, B. Bollobás (ed.), Cambridge University Press, (1986), 200 pp.

R. Rado, A sequence of polyhedra having intersections of specified dimensions, *J. London Math. Soc.* **22** (1947), 287–289.

H. Tietze, Über Nachbargebiete in Raume, *Monatsch. Math.* **16** (1905), 211–216.

## 129. Neighbourly Simplices: Perles' Theorem

*There are at most  $2^{n+1}$  neighbourly  $n$ -simplices in  $\mathbb{R}^n$ .*

*Proof.* Let  $S_1, \dots, S_p$  be neighbourly  $n$ -simplices in  $\mathbb{R}^n$ , so that our task is to prove that  $p \leq 2^{n+1}$ . Let  $H_1, \dots, H_q$  be the all the hyperplanes determined by the facets of these simplices, and write  $H_j^+$  and  $H_j^-$  for the two closed half-spaces determined by  $H_j$ . Define a  $p \times q$  matrix  $B = (b_{ij})$  by setting

$$b_{ij} = \begin{cases} 0, & \text{if } S_i \text{ has no facets on } H_j, \\ 1, & \text{if } S_i \text{ has a facet on } H_j \text{ and } S_i \subset H_j^+ \\ -1, & \text{if } S_i \text{ has a facet on } H_j \text{ and } S_i \subset H_j^-. \end{cases}$$

Note that every row of  $B$  has  $n + 1$  non-zero entries (since every simplex  $S_i$  has  $n + 1$  facets) and  $p - n - 1$  zero entries. Also, for any two rows there is a column in which one is 1 and the other  $-1$ , i.e., if  $1 \leq i < j \leq p$  then  $b_{ih}b_{jh} = -1$  for some  $h$ :  $b_{ih} = -b_{jh} \neq 0$ .

Construct a  $\pm 1$  matrix  $C$  with  $p \times 2^{q-n-1}$  rows and  $q$  columns by replacing each row  $\mathbf{b}_i$  of  $B$  with the  $2^{q-n-1}$  different row vectors obtained from  $\mathbf{b}_i$  by changing each 0 entry into 1 or  $-1$ . Now, every two rows of  $C$  are different since either they come from different row vectors of  $B$ , so they differ in a non-zero entry, or they are different replacements of the same row vector. But there are only  $2^q$  different vectors of length  $q$  and entries  $\pm 1$ , so  $p \cdot 2^{q-n-1} \leq 2^q$ , i.e.,  $p \leq 2^{n+1}$ , as required.  $\square$

**Notes.** The result and the beautiful proof above are due to Perles.

Let us write  $f(n)$  for the maximal number of neighbourly simplices in  $\mathbb{R}^n$ . Bagemihl proved that  $8 \leq f(3) \leq 17$ , and later Baston showed that  $f(3) \leq 9$ . Baston also conjectured that  $f(n) = 2^n$  for every  $n$ : this conjecture seems to be still open. After numerous lower and upper bounds on  $f(n)$ , Perles proved the assertion in this question that  $f(n) \leq 2^{n+1}$ , and, simultaneously, Zaks showed that  $f(n) \geq 2^n$  for every  $n$ .

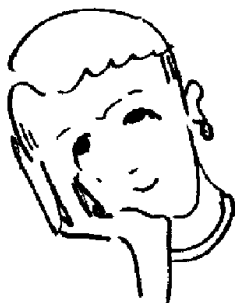
F. Bagemihl, Neighbouring tetrahedra, *Amer. Math. Monthly* **63** (1956), 328–329.

V.J.D. Baston, *Some Properties of Polyhedra in Euclidean Space*, Pergamon Press, (1965).

M.A. Perles, At most  $2^{d+1}$  simplices in  $\mathbb{E}^d$ , in *Convexity and Graph Theory* (Jerusalem, 1981), North-Holland Math. Stud. **87**, North-Holland, (1984), pp. 253–254.

J. Zaks, Neighborly families of  $2^d$  simplices in  $\mathbb{E}^d$ , *Geom. Dedicata* **11** (1981), 505–507.

J. Zaks, Arbitrarily large families of symmetric convex polytopes, *Geom. Dedicata* **20** (1986), 175–179.



### 130. The Rank of a Matrix

For  $i = 0, 1, \dots, n$ , let  $A_1^i, A_2^i, \dots, A_{\binom{n}{i}}^i$  be an enumeration of the  $i$ -subsets of  $[n]$ . Also, for  $1 \leq i \leq j \leq n$ , define an  $\binom{n}{i}$  by  $\binom{n}{j}$  matrix  $N(i, j) = N(i, j; n)$  by

$$N(i, j)_{s,t} = \begin{cases} 1 & \text{if } A_s^i \subset A_t^j, \\ 0 & \text{otherwise.} \end{cases}$$

Then the matrix  $N(i, j)$  has full rank, i.e.,

$$\text{rank } N(i, j) = \min \left\{ \binom{n}{i}, \binom{n}{j} \right\}.$$

*Proof.* If we replace a set by its complement in  $[n]$ , so that  $i$ -sets become  $(n-i)$ -sets and  $j$ -sets  $(n-j)$ -sets, then the containments are reversed, and  $N(i, j)$  goes into its transpose. Therefore we may assume that  $0 \leq i \leq j \leq n-i$ , so that  $\binom{n}{i} \leq \binom{n}{j}$ .

Set  $\mathcal{A} = [n]^{(i)}$  and  $\mathcal{B} = [n]^{(j)}$ , and let  $\{e_A : A \in \mathcal{A}\}$  be the standard basis of  $\mathbb{R}^{\binom{n}{i}}$  indexed with the elements of  $\mathcal{A}$ . Furthermore, for  $B \in \mathcal{B}$ , set

$$f_B = \sum \{e_A : A \in \mathcal{A}, A \subset B\},$$

so that the vectors  $f_B$  are the column vectors of  $N(i, j)$ . Thus, our task is to prove that the vectors  $f_B$ ,  $B \in \mathcal{B}$ , span  $\mathbb{R}^{\binom{n}{i}}$ , i.e., each  $e_A$ ,  $A \in \mathcal{A}$ , is a linear combination of the  $f_B$ .

Let us fix a set  $A_0 \in \mathcal{A}$ . For  $t = 0, 1, \dots, i$ , put

$$g_t = \sum \{e_{A'} : A' \in \mathcal{A}, |A_0 \setminus A'| = t\},$$

so that  $e_{A_0} = g_0$ , and

$$h_t = \sum \{f_B : B \in \mathcal{B}, |A_0 \setminus B| = t\}.$$

Since  $i \leq j \leq n-i$ , the vectors  $g_t$ ,  $h_t$  are sums of non-empty collections of vectors.

Expanding  $f_B$  and so expressing  $h_t$  in the basis  $e_A$ ,  $A \in \mathcal{A}$ ,

$$h_t = \sum_{A \in \mathcal{A}} c_A e_A,$$

note that  $c_A = 0$  if  $|A_0 \setminus A| < t$ , and if  $|A_0 \setminus A| = |A_0 \setminus A'|$  then  $c_A = c_{A'}$ .

Consequently,

$$h_t = \sum_{s=t}^i c_{st} g_s$$

for some constants  $c_{st}$  with  $c_{tt} \neq 0$  (in fact,  $c_{tt} > 0$ ) for  $t = 0, 1, \dots, i$ . Since the  $(i+1) \times (i+1)$  matrix  $(c_{st})$  is upper-triangular, with non-zero entries on the diagonal, it is invertible, and so

$$g_0 = \sum_{t=0}^i d_t h_t$$

for some constants  $d_0, \dots, d_i$ . □

**Notes.** For  $j = n - i$  the square matrix  $N(i, j)$  is just the adjacency matrix of the Kneser graph on the set  $[n]^{(i)}$  of  $i$ -sets of  $[n]$  in which two  $i$ -sets are joined if and only if they are disjoint (see Problem 118). In fact, the non-singularity of this adjacency matrix is rather trivially equivalent to the more general assertion in this problem.

When, from the early 1970s, I set this question in my Algebra course in Cambridge, most students found it rather difficult.

### 131. Modular Intersecting $k$ -uniform Set Systems: a Theorem of Frankl and Wilson

Let  $p$  be a prime and  $1 < k < n$ ; furthermore, let  $\mu_0, \mu_1, \dots, \mu_r$  be distinct residues modulo  $p$ , with  $k \equiv \mu_0 \pmod{p}$ . Let  $\mathcal{F}$  be a collection of  $k$ -subsets of  $[n]$  such that if  $A$  and  $B$  are distinct elements of  $\mathcal{F}$  then there is an index  $i$ ,  $1 \leq i \leq r$ , such that

$$|A \cap B| \equiv \mu_i \pmod{p},$$

and every  $\mu_i$  occurs in one of these relations. Then

$$|\mathcal{F}| \leq \binom{n}{r}.$$

*Proof.* Before we turn to the proof, we shall make use of the matrices  $N(i, j)$  defined in Problem 130. Thus, for every  $i$ ,  $0 \leq i \leq n$ , fix an enumeration  $A_1^i, A_2^i, \dots, A_{\binom{n}{i}}^i$  of all the  $i$ -subsets of  $[n]$ . Also, for  $1 \leq i \leq j \leq n$ , define an  $\binom{n}{i}$  by  $\binom{n}{j}$  matrix  $N(i, j)$  encoding the containments  $A_s^i \subset A_t^j$ , i.e., let the  $(s, t)$ -entry of  $N(i, j)$  be

$$N(i, j)_{s,t} = \begin{cases} 1 & \text{if } A_s^i \subset A_t^j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $N(i, j)$  is a 0-1 matrix; for  $i = j$  it is the identity matrix (and for  $i > j$  it would be the identically zero matrix). Also for  $i = 0$ , the matrix  $N(0, j)$  is just the  $\binom{n}{j}$ -dimensional vector  $(1, 1, \dots, 1)$ .

Let  $v_1, v_2, \dots, v_{\binom{n}{i}}$  be the row-vectors of  $N(i, j)$ , and denote by  $V_{ij}$  the subspace of  $\mathbb{R}^{\binom{n}{j}}$  they span. We shall need only the trivial inequality

$$\dim(V_{ij}) \leq \binom{n}{i},$$

although from Problem 130 we know that

$$\dim(V_{ij}) = \text{rank } N(i, j) = \min \left\{ \binom{n}{i}, \binom{n}{j} \right\}.$$

Furthermore, let  $M(i, j)$  be the  $\binom{n}{j} \times \binom{n}{j}$  matrix in which the  $(s, t)$  entry is

$$M(i, j)_{s,t} = \binom{|A_s^j \cap A_t^j|}{i},$$

and let  $W_{ij}$  be the subspace of  $\mathbb{R}^{\binom{n}{j}}$  spanned the row vectors of this matrix  $M(i, j)$ .

We gather some easy but important properties of these matrices and vector spaces into the following Claim.

**Claim.** *Let  $0 \leq i \leq j \leq k \leq n$ . Then*

$$W_{ik} \subset V_{ik} \subset V_{jk}.$$

*In particular, if*

$$M = \sum_{i=0}^j a_i M(i, k),$$

*then*

$$\text{rank } M \leq \dim(V_{jk}) \leq \binom{n}{j}.$$

*Proof of the Claim.* Given sets  $A_s^i \subset A_u^k$ , there are  $\binom{k-i}{j-i}$  sets  $A_t^j$  with  $A_s^i \subset A_t^j \subset A_u^k$ , so

$$N(i, j)N(j, k) = \binom{k-i}{j-i} N(i, k).$$

In particular, each row vector of  $N(i, k)$  is a linear combination of the row vectors of  $N(j, k)$ , i.e.,  $V_{ik} \subset V_{jk}$ .

Furthermore,

$$M(i, k) = N(i, k)^T N(i, k),$$

where  $N(i, k)^T$  is the transpose of  $N(i, k)$ . Indeed, the product of the  $s$ th row of  $N(i, k)^T$  (i.e., the  $s$ th column of  $N(i, k)$ ) and the  $t$ th column of  $N(i, k)$  is the number of  $i$ -subsets contained in  $A_s^k$  and  $A_t^k$ , i.e.,  $\binom{|A_s^k \cap A_t^k|}{i}$ . Consequently, the row vectors of  $M(i, k)$  are linear combinations of the row vectors of  $N(i, k)$ , implying that  $W_{ik} \subset V_{ik}$ , as claimed.  $\square$

Having established this Claim, which has nothing to do with our set system  $\mathcal{F}$ , let us turn to the proof of the assertion. We may assume that  $r < k$  since otherwise there is nothing to prove. Our aim is to find a linear combination  $M$  of the matrices



$M(0, k), M(1, k), \dots, M(r, k)$  which has rank at least  $|\mathcal{F}|$ ; then we are done by the Claim, since then

$$|\mathcal{F}| = \text{rank } M \leq \dim(V_{rk}) \leq \binom{n}{r}.$$

To this end, choose integers  $0 \leq a_i < p$  for  $0 \leq i \leq r$  such that for every integer  $x$  we have

$$\prod_{i=1}^r (x - \mu_i) \equiv \sum_{i=0}^r a_i \binom{x}{i} \pmod{p}.$$

This can be done by first choosing  $a_r$ , then  $a_{r-1}$ , and so on. Having chosen these  $a_i$ , set

$$M = \sum_{i=0}^r a_i M(i, k).$$

Now, let us use  $\mathcal{F}$  to give a lower bound for rank  $M$ : let  $M(\mathcal{F})$  be the  $|\mathcal{F}| \times |\mathcal{F}|$  minor of  $M$  formed by the entries  $M_{s,t}$  with  $A_s^k, A_t^k \in \mathcal{F}$ . We claim that, modulo  $p$ , the matrix  $M(\mathcal{F})$  is a non-zero multiple of the identity matrix, so has full rank  $|\mathcal{F}|$ .

Indeed, if  $s \neq t$  then

$$M_{s,t} = \sum_{i=0}^r a_i \binom{|A_s^k \cap A_t^k|}{i} \equiv \sum_{i=0}^r a_i \binom{\mu_j}{i} \equiv 0 \pmod{p},$$

where  $|A_s^k \cap A_t^k| \equiv \mu_j$ ,  $1 \leq j \leq r$ . Also,

$$M_{s,s} = \sum_{i=0}^r a_i \binom{|A_s^k|}{i} \equiv \sum_{i=0}^r a_i \binom{k}{i} \not\equiv 0 \pmod{p},$$

since, by our choice of the  $a_i$ ,

$$\sum_{i=0}^r a_i \binom{k}{i} \equiv \prod_{i=1}^r (k - \mu_i) \not\equiv 0.$$

Consequently, modulo  $p$  the matrix  $M(\mathcal{F})$  is indeed a non-zero multiple of the  $|\mathcal{F}| \times |\mathcal{F}|$  identity matrix, so

$$|\mathcal{F}| = \text{rank } M(\mathcal{F}) \leq \text{rank } M \leq \dim(V) \leq \binom{n}{r},$$

completing our proof. □

**Notes.** This striking result is the first of many beautiful results from P. Frankl and R.M. Wilson, Intersection theorems with geometric consequences, *Combinatorica* **1** (1981), 357–368.

### 132. Families without Orthogonal Vectors

Let  $p$  be a prime, and let  $X \subset \{1, -1\}^{4p} \subset \mathbb{R}^{4p}$  be a family of balanced vectors no two of which are orthogonal. Then

$$|X| \leq 2 \binom{4p-1}{p-1}.$$

*Proof.* Identify each vector  $x = (x_1, \dots, x_{4p}) \in X$  with the set  $S_x = \{i : x_i = 1\}$ ; this gives us a family  $\mathcal{F} \subset [4p]^{(2p)}$  of  $2p$ -subsets of  $[4p]$  such that  $|S_x \cap S_y| \neq p$  for  $S_x, S_y \in \mathcal{F}$ , i.e.,  $x, y \in X$ . Note that some element  $i \in [4p]$  is in at least  $2p|\mathcal{F}|/4p \geq |\mathcal{F}|/2$  sets belonging to  $\mathcal{F}$ . Therefore, writing

$$\mathcal{F}_0 = \{A \subset [4p-1] : A \cup \{4p\} \in \mathcal{F}\},$$

we may assume that  $|\mathcal{F}_0| \geq |\mathcal{F}|/2$ . In view of this, it suffices to prove the following assertion.

Let  $p$  be a prime, and set  $n = 4p - 1$  and  $k = 2p - 1$ . Let  $\mathcal{F}_0 \subset [n]^{(k)}$  be a family of  $k$ -subsets of  $[n]$  such that if  $A, B \in \mathcal{F}_0$  then  $|A \cap B| \neq p - 1$ . Then

$$|\mathcal{F}_0| \leq \binom{4p-1}{p-1}. \quad (54)$$

The proof of inequality (54) we shall present is similar to that of Problem 131. Let

$$M = M(p-1, k),$$

be the  $\binom{n}{k} \times \binom{n}{k}$  matrix defined in Problem 131:

$$M_{s,t} = \binom{|A_s^k \cap A_t^k|}{p-1},$$

where  $A_1^k, A_2^k, \dots, A_{\binom{n}{k}}^k$  is an enumeration of the  $k$ -subsets of  $[n]$ . We know from the Claim in the solution to Problem 131 that

$$\text{rank } M \leq \binom{n}{p-1}.$$

Consider the minor  $F$  of  $M$  corresponding to  $\mathcal{F}_0$ . Since  $\binom{a}{p-1}$  is not divisible by  $p$  if and only if  $a \equiv -1 \pmod{p}$ , an entry  $F_{st}$  of  $F$  is non-zero modulo  $p$  if and only if  $|A_s^k \cap A_t^k|$  is  $2p-1$  or  $p-1$ . The former happens if and only if  $s = t$ , and the latter never happens since the main condition in our assertion is precisely that  $|A_s^k \cap A_t^k| \neq p-1$  for  $A_s^k, A_t^k \in \mathcal{F}_0$ .

Therefore an entry  $F_{st}$  of  $F$  is non-zero modulo  $p$  if and only if it is on the diagonal of  $F$ . *A fortiori*,  $F$  has full rank, and so

$$|\mathcal{F}_0| = \text{rank } F \leq \text{rank } M \leq \binom{n}{p-1},$$

completing the proof of inequality (54). □

**Notes.** This is another beautiful result from P. Frankl and R.M. Wilson: Intersection theorems with geometric consequences, *Combinatorica* **1** (1981), 357–368. In fact, Frankl and Wilson show this assertion for prime powers, rather than primes. To prove this, all we have to notice is that the trivial divisibility condition holds in this case as well: if  $q = p^\alpha$  then  $p$  divides  $\binom{a}{q-1}$  if and only if  $a \not\equiv -1 \pmod{q}$ .

### 133. A Counterexample to Borsuk's Conjecture: the Kahn–Kalai Theorem

Let  $k(n)$  be the minimal integer such that if a set  $K \subset \mathbb{R}^n$  has diameter 1 then  $K$  is the union of  $k(n)$  sets of diameter less than 1. Then, if  $n$  is large enough,

$$k(n) \geq c\sqrt{n}$$

for some  $c > 1$ .

*Proof.* In fact, we shall prove more. An  $(n, k, t)$ -family is a  $t$ -intersecting  $k$ -uniform hypergraph  $(V, \mathcal{F})$  with  $n$  vertices. Thus,  $\mathcal{F}$  is a family of  $k$ -subsets of  $V = [n]$ , say, such that  $|A \cap B| \geq t$  for all  $A, B \in \mathcal{F}$ . Let  $h_{k,t}(n)$  be the minimal integer such that every  $(n, k, t)$ -family is the union of  $h_{k,t}(n)$  number of  $(n, k, t+1)$ -families, and set  $h(n) = \max_{k,t} h_{k,t}(n)$ . Identifying a set  $A \in \mathcal{F}$  with its characteristic vector  $x^A = (x_i^A)_{i=1}^n$ , where

$$x_i^A = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{otherwise,} \end{cases}$$

we see that  $h(n) \leq k(n)$  for every  $n$ . We shall show that

$$h(n) \geq (1.2)\sqrt{n} \tag{55}$$

if  $n$  is large enough.

In fact, we shall show that the Frankl–Wilson theorem in Problem 132 implies that

$$h(n) \geq (1.203)\sqrt{n} \tag{56}$$

when  $n = 2p(4p - 1)$  for a prime  $p$ ; then (55) is immediate from the fact that, if  $p = p(n)$  is the largest prime such that  $2p(4p - 1) \leq n$ , then  $n - 2p(4p - 1) = o(n)$ .

Let then  $p$  be a prime, and let  $W = [4p]^{(2)}$  be the set of unordered pairs of elements of  $W$ , i.e., the edge set of the complete graph with vertex set  $W$ . Thus  $|W| = n = 2p(4p - 1)$ . For a  $2p$ -set  $A$  of  $V = [4p]$ , let  $F_A$  be the set of unordered pairs of elements of  $V$  with precisely one element in  $A$ , i.e., the edge set of the complete bipartite graph with bipartition  $(A, [4p] \setminus A)$ . Thus  $F_A$  is a  $4p^2$ -element subset of  $W$ . Let  $\mathcal{F}$  be the collection of these sets  $F_A$ :

$$\mathcal{F} = \{F_A : A \in [4p]^{(2p)}, 1 \in A\},$$

so that

$$|F_A| = \frac{1}{2} \binom{4p}{2p}.$$

Note that if  $F_A, F_B \in \mathcal{F}$  then

$$|F_A \cap F_B| \geq 2p^2,$$

with equality if and only if  $|A \cap B| = p$ . Thus  $\mathcal{F}$  is an  $(n, k, t)$ -family with  $n = 2p(4p - 1)$ ,  $k = 4p^2$  and  $t = 2p^2$ . Furthermore, if  $\mathcal{F}' \subset \mathcal{F}$  is an  $(n, k, t + 1)$ -family then  $|A \cap B| \neq p$  for  $F_A, F_B \in \mathcal{F}'$ . Therefore, by the Frankl–Wilson theorem in Problem 132, if  $\mathcal{F}' \subset \mathcal{F}$  is an  $(n, k, t + 1)$ -family then

$$|\mathcal{F}'| \leq 2 \binom{4p-1}{p-1}.$$

Consequently,

$$h(n) \geq \frac{\frac{1}{2} \binom{4p}{2p}}{2 \binom{4p-1}{p-1}} = \frac{\binom{4p}{2p}}{16 \binom{4p}{p}} \geq \frac{1}{32} 2^{4p/4p} (4/3)^{3p} = \frac{1}{32} (27/16)^p > (1.203)^{\sqrt{n}},$$

if  $n = 2p(4p - 1)$  is large enough.  $\square$

**Notes.** As we remarked in Problem 42, in 1933 Borsuk posed the problem of whether every set of diameter 1 in  $\mathbb{R}^n$  is the union of  $n + 1$  sets of diameter less than 1. Later the problem became well known as Borsuk's conjecture. Although in the 1980s Erdős and Larman raised the possibility of this 'conjecture' being false, it was a great surprise to most mathematicians familiar with Borsuk's conjecture when Kahn and Kalai gave the beautiful and surprisingly simple counterexample we presented above. The bound

$$h(n) \geq \frac{\binom{4p}{2p}}{\binom{4p}{p}}$$

shown above implies that Borsuk's conjecture is false for  $n = 1, 325$  and every  $n > 2, 014$ . A little later Nilli (a.k.a. Noga Alon) improved the Kahn–Kalai construction to show that for  $n = 946$  there are subsets of  $\mathbb{R}^n$  that cannot be partitioned into at most  $n + 1$  sets of strictly smaller diameter. Further improvements have been given by many people, including Raĭgorodskiĭ (to 561), Weissbach (to 560), Hinrichs (to 323), Hinrichs and Richter (to 298). All the constructions have been inspired by the Kahn–Kalai breakthrough, with some new ingredients added. For example, Hinrichs made use of vectors of minimal lengths in the Leech lattice to construct a finite set in  $\mathbb{R}^{323}$  that is not the union of 561 sets of smaller diameter. For the history of Borsuk's problem see Raĭgorodskiĭ's article celebrating the seventieth anniversary of the problem.

K. Borsuk, Drei Sätze über die  $n$ -dimensionale euklidische Sphäre, *Fund. Math.* **20** (1933), 177–190.

P. Erdős, My Scottish book 'problems', in *The Scottish Book – Mathematics from the Scottish Café* (R.D. Mauldin, ed.), Birkhäuser, (1981), pp. 35–43.

- A. Hinrichs, Spherical codes and Borsuk's conjecture, *Discrete Math.* **243** (2002), 253–256.
- A. Hinrichs and C. Richter, New sets with large Borsuk numbers, *Discrete Math.* **270** (2003), 137–147.
- J. Kahn and G. Kalai, A counterexample to Borsuk's conjecture, *Bull. Amer. Math. Soc.* **29** (1993), 60–62.
- D. Larman, Open Problem 6, in *Convexity and Graph Theory*, M. Rosenfeld and J. Zaks (eds), *Ann. Discrete Math.* **20**, North-Holland, (1984), p. 336.
- A. Nilli, On Borsuk's problem, in *Contemporary Mathematics* **178** (1994), 209–210.
- A.M. Raĭgorodskiĭ, On dimensionality in the Borsuk problem, *Uspekhi Mat. Nauk* **52** (1997), 181–182 (in Russian); translation in *Russian Math. Surveys* **52** (1997), 1324–1325.
- A.M. Raĭgorodskiĭ, On a bound in the Borsuk problem, *Uspekhi Mat. Nauk* **54** (1999), 185–186 (in Russian); translation in *Russian Math. Surveys* **54** (1999), 453–454.
- A.M. Raĭgorodskiĭ, The Borsuk partition problem: the seventieth anniversary, *Mathematical Intelligencer* **26** (2004), 4–12.
- A.M. Raĭgorodskiĭ and A. Chernov, A., The Borsuk problem for some classes of  $(0, 1)$ -polytopes, *Chebyshevskii Sb.* **5** (2004), 98–102 (in Russian).
- B. Weissbach, Sets with large Borsuk number, *Beiträge Algebra Geom.* **41** (2000), 417–423.

### 134. Periodic Sequences

Let  $(a_n)_0^\infty, (b_n)_0^\infty$  be periodic sequences of periods  $p$  and  $q$ , respectively. If  $a_n = b_n$  for  $p + q - (p, q)$  consecutive integers  $n$  then the two sequences are identical.

*Proof.* We assume, as we may, that  $a_n$  and  $b_n$  are real numbers, and use them to define formal power series  $F(X)$  and  $G(X)$ . Also, we may assume that  $a_n = b_n$  for  $n = 0, 1, \dots, p + q - (p, q)$ , since if  $a_n = b_n$  for  $n \geq n_0$  then the periodicity of  $(a_n)$  and  $(b_n)$  implies that  $a_n = b_n$  for all  $n$ .

Since  $(a_n)$  has period  $p$ ,

$$F(X) = \sum_{n=0}^{\infty} a_n X^n = (1 - X^p)^{-1} P(X)$$

for some polynomial  $P(X)$  of degree at most  $p - 1$ . Similarly,

$$G(X) = \sum_{n=0}^{\infty} b_n X^n = (1 - X^q)^{-1} Q(X)$$

for some polynomial  $Q(X)$  of degree at most  $q - 1$ .

The polynomial  $1 - X^{(p,q)}$  divides both  $1 - X^p$  and  $1 - X^q$ , so

$$D(X) = F(X) - G(X) = (1 - X^{(p,q)})(1 - X^p)^{-1}(1 - X^q)^{-1}R(X),$$

where  $R(X)$  is a polynomial of degree at most  $p + q - (p, q) - 1$ . Since the first  $p + q - (p, q)$  coefficients of  $D(X)$  are 0, the polynomial  $R(X)$  is identically 0; therefore, the power series  $D(X)$  is identically 0, completing the solution.  $\square$

**Notes.** The result above is due to Fine and Wilf; the short and elegant proof above was given by Ernst Straus. Fine and Wilf use this result to deduce the following two theorems about periodic functions.

**I.** Let  $f(x)$  and  $g(x)$  be continuous periodic functions on  $\mathbb{R}$  of periods  $\alpha$  and  $\beta$ , respectively, where  $\alpha/\beta = p/q$ ,  $(p, q) = 1$ , is rational. If  $f(x) = g(x)$  on an interval of length  $\alpha + \beta - \beta q^{-1}$ , then  $f(x) \equiv g(x)$ . The result would be false if  $\alpha + \beta - \beta q^{-1}$  were replaced by anything smaller.

**II.** Let  $f(x)$  and  $g(x)$  be continuous periodic functions on  $\mathbb{R}$  of periods  $\alpha$  and  $\beta$ , respectively, where  $\alpha/\beta$  is irrational. If  $f(x) = g(x)$  on an interval of length  $\alpha + \beta$ , then  $f(x) \equiv g(x)$ . The result would be false if  $\alpha + \beta$  were replaced by anything smaller.

N.J. Fine and H.S. Wilf, Uniqueness theorems for periodic functions, *Proc. Amer. Math. Soc.* **16** (1965), 109–114.



### 135. Periodic Words: the Fine–Wilf Theorem

Let  $p < q$  be relatively prime natural numbers, and let  $w$  be a word of length  $n$  which is  $p$ -periodic and  $q$ -periodic. If  $n \geq p + q - 1$  then  $w$  is constant, but this need not be the case if  $n < p + q - 1$ .

*First Proof.* We shall give three proofs: in the first two we shall prove the first assertion only, and leave the proof of the second (very easy) assertion to the third proof, where it arises without any additional work.

Thus, assume that  $n \geq p + q - 1$ . Let  $u = u_1 u_2 \dots$  be the infinite word such that  $u_i = w_j$  if  $i \equiv j \pmod{p}$ ; similarly, let  $v = v_1 v_2 \dots$  be given by  $v_i = w_j$  if  $i \equiv j \pmod{q}$ . Since  $p$  and  $q$  are periods of  $w$ , the words  $u$  and  $v$  are well defined. By construction,  $u$  is  $p$ -periodic and  $v$  is  $q$ -periodic, and they agree in their first  $n \geq p + q - 1$  letters. Hence, by the Fine–Wilf theorem in Problem 134, the infinite words  $u$  and  $v$  coincide:  $u_i = v_i$  for every  $i$ . This means exactly that  $u$  is both  $p$ -periodic and  $q$ -periodic. Since  $p$  and  $q$  are relatively prime, there are natural numbers  $s$  and  $t$  such that  $ps - qt = 1$ . Hence,

$$u_{i+1} = u_{i+qt+1} = u_{i+ps} = u_i$$

for every  $i$ , i.e.,  $u$  is constant, and so is its initial segment  $w$ . □

*Second Proof.* Since  $(p, q) = 1$ , there is an enumeration  $i_1, i_2, \dots, i_p$  of the integers  $1, 2, \dots, p$  such that  $i_j + q \equiv i_{j+1} \pmod{p}$  for  $i = 1, \dots, p-1$ . We claim that  $w_{i_1} = w_{i_2} = \dots = w_{i_p}$ : as  $w$  is  $p$ -periodic, this implies that it is constant. To prove this assertion, first note that for  $1 \leq j < p$  we have  $1 \leq i_j < p$ . Since  $w$  has period  $p$ , our word is constant on congruence classes modulo  $p$ :  $w_{i_j} = w_{i_j+p} = \dots = 0$ , say. As  $q$  is also a period of  $w$  and  $i_j + q \leq p + q - 1$ , we have  $w_{i_j+q} = 0$  as well. Hence, the  $p$ -periodicity of  $w$  and the congruence  $i_j + q \equiv i_{j+1} \pmod{p}$  imply that  $w_{i_{j+1}} = 0$ , completing the proof. □

*Third Proof.* Let  $G_{n;p,q}$  be the graph with vertex set  $[n] = \{1, 2, \dots, n\}$  such that for  $1 \leq i < j \leq n$  the vertices  $i$  and  $j$  are joined by an edge if and only if  $j - i$  is  $p$  or  $q$ . We call an edge  $ij$  with  $i < j$  a  $p$ -edge if  $j - i = p$ , and a  $q$ -edge if  $j - i = q$ . The condition that  $p$  and  $q$  are periods of  $w$  is equivalent to the assertion that  $w$  is constant on the components of  $G_{n;p,q}$  (i.e.,  $w_i = w_j$  if  $i$  and  $j$  belong to the same component of  $G$ ). Hence our task is to show that  $G_{n;p,q}$  is connected if and only if  $n \geq p + q - 1$ .

Suppose that  $n \leq p + q - 1$ . We shall prove more than is needed:  $G_{n;p,q}$  is a forest, and it is a tree if and only if  $n = p + q - 1$ . To see this, note that every vertex

of  $G_{n;p,q}$  has degree at most two, and no vertex is incident with two  $q$ -edges. Even more, if a vertex  $i$  is joined to  $j$  by a  $p$ -edge and to  $k$  by a  $q$ -edge, then either both  $j$  and  $k$  are larger than  $i$ , or else both are smaller. Hence, a cycle in  $G_{n;p,q}$  would have to be of the form  $i_1 j_1 i_2 j_2 \dots i_s j_s$ , where  $i_h j_h$  is a  $q$ -edge with  $j_h - i_h = q$ , and from  $j_h$  to  $i_{h+1} \pmod s$  the arc of the cycle is a decreasing sequence of edges. Hence,  $sq = tp$  for some natural number  $t$  with  $2s \leq s + t \leq n$ . But then  $q$  divides  $t$ , so  $t \geq q$ ; similarly,  $s \geq p$ . However, this gives the contradiction  $p + q \leq s + t \leq n \leq p + q - 1$ . Hence  $G_{n;p,q}$  is indeed a forest.

What are the degrees in  $G_{n;p,q}$ ? If  $n = p + q - 1$  then each of  $p$  and  $q$  has degree 1, and every other vertex has degree 2, so  $G_{n;p,q}$  is a path. If  $n \leq p + q - 2$  then  $p - 1$ ,  $p$ ,  $q - 1$  and  $q$  have degree one, and the other vertices have degree two, so  $G_{n;p,q}$  is the union of two paths. □

**Notes.** The result in this question tends to be called the Fine–Wilf theorem: clearly, it is a little stronger than its form in Problem 134. In fact, it is usually stated for arbitrary periods as follows.

*Let  $w$  be a word such that  $p$  and  $q$  are periods of  $w$ , but  $(p, q)$  is not a period. Then  $w$  has length at most  $p + q - (p, q) - 1$ .*

The deduction of this from the special case  $(p, q) = 1$  is trivial.

### 136. Points on a Hemisphere: Wendel's Theorem

Choose  $n$  points at random on the surface of the unit sphere in  $\mathbb{R}^d$  and write  $p_{d,n}$  for the probability that all  $n$  points lie on some hemisphere. Then

$$p_{d,n} = 2^{-n+1} \sum_{k=0}^{d-1} \binom{n-1}{k}.$$

*Proof.* Given points  $x_1, \dots, x_n \in S^{d-1} \subset \mathbb{R}^d$ , for  $i = 1, \dots, n$ , let  $X_i$  be the hyperplane in  $\mathbb{R}^d$  orthogonal to  $x_i$ . We call a connected component of  $\mathbb{R}^d \setminus \bigcup_{i=1}^n X_i$  a *compartment* of  $\mathbb{R}^d$  determined by  $\mathbf{x} = (x_1, \dots, x_n)$ . Putting it another way, define the *sign* of a point  $y \in S^{d-1}$  to be  $\sigma(x_1, \dots, x_n; y) = (\sigma_1, \dots, \sigma_n)$ , where  $\sigma_i$  is the sign of the product  $\langle x_i, y \rangle$ . The (inner) product of two vectors  $a = (a_1, \dots, a_n)$  and  $b = (b_1, \dots, b_n)$  is  $\langle a, b \rangle = \sum_{i=1}^n a_i b_i$ , and the sign of a real number  $t \in \mathbb{R}$  is

$$\text{sign } t = \begin{cases} 1 & \text{if } t > 0, \\ -1 & \text{if } t < 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Clearly, for almost all choices of  $x_1, \dots, x_n$  and  $y$ , the sign  $\sigma(y) = \sigma(x_1, \dots, x_n; y)$  takes one of  $2^n$  values, namely  $\sigma(y) \in \{1, -1\}^n$ . Then a compartment of  $\mathbb{R}^d \setminus \bigcup_{i=1}^n X_i$  consists of the points  $y \in S^{d-1}$  with  $\sigma(y) = s$  for some fixed value  $s \in \{1, -1\}^n$ . We call  $s$  the *sign* of this compartment. Note that there need not exist a compartment with a given sign  $s \in \{1, -1\}^n$ ; putting it slightly imprecisely, the compartment with sign  $s$  need not be non-empty.

The probability that there is a hemisphere that contains all the  $n$  points  $x_1, \dots, x_n$  is precisely the probability that the compartment with  $\sigma(y) = (1, \dots, 1)$  is not empty, namely

$$p_{d,n} = \mathbb{P}(\sigma(x_1, \dots, x_n; y) = (1, \dots, 1) \text{ for some } y \in S^{d-1}).$$

Let us note that all  $2^n$  signs have the same probability. Indeed, if we fix  $y$  and write  $r_y(x_i)$  for the reflection of  $x_i$  in the plane orthogonal to  $y$ , i.e.,  $r_y(x_i) = x_i - 2\langle x_i, y \rangle y$ , then

$$\sigma(x_1, \dots, x_{i-1}, r(x_i), x_{i+1}, \dots, x_n; y) = (\sigma_1, \dots, \sigma_{i-1}, -\sigma_i, \sigma_{i+1}, \dots, \sigma_n),$$

so the signs  $(\sigma_1, \dots, \sigma_i, \dots, \sigma_n)$  and  $(\sigma_1, \dots, -\sigma_i, \dots, \sigma_n)$  have the same probability. Consequently all signs have the same probability.

The heart of the proof is the fact that if the points  $x_1, \dots, x_n \in S^{d-1}$ ,  $n \geq 1$ , are in general position, i.e., any  $d$  of them span  $\mathbb{R}^d$ , then  $\mathbf{x}$  has some  $N(d, n)$  compartments, i.e., the number of compartments is *independent* of the actual positions of the points  $x_1, \dots, x_n$ .

We shall prove this fact by induction on  $n$ . For  $n = 1$  the assertion is clear, namely  $N(d, 1) = 2$ , so let us pass on to the proof of the induction step. By the induction hypothesis,  $x_1, \dots, x_{n-1}$  determine  $N(d, n-1)$  compartments. After the addition of the point  $x_n$  or, equivalently, the plane  $X_n$ , some of the compartments remain the same, while some others are divided into two by  $X_n$ . Hence  $N(d, n) = N(d, n-1) + N^*$ , where  $N^*$  is the number of the compartments divided by  $X_n$ . Equivalently,  $N^*$  is the number of compartments of the hyperplane  $X_n$  determined by the  $(d-2)$ -dimensional planes  $X_1 \cap X_n, X_2 \cap X_n, \dots, X_{n-1} \cap X_n$ .

By the induction hypothesis,  $N^* = N(d-1, n-1)$ , and so

$$N(d, n) = N(d, n-1) + N(d-1, n-1) \quad (57)$$

is indeed independent of the actual position of the points  $x_1, \dots, x_n$ . Furthermore, since  $N(1, n) = N(d, 1) = 2$  for all  $d, n \geq 1$ , relation (57) implies by induction on  $d + n$  that

$$N(d, n) = 2 \sum_{k=0}^{d-1} \binom{n-1}{k}.$$

All that remains is to draw the appropriate conclusions. For almost every choice of  $x_1, \dots, x_n \in S^{d-1}$ , the sequence  $\mathbf{x} = (x_1, \dots, x_n)$  determines the same  $N(d, n)$  number of compartments. Also, the probability that for a sequence  $\mathbf{x}$  there is a compartment of sign  $\sigma \in \{1, -1\}^n$  is independent of  $\sigma$ . Hence, for  $\sigma \in \{1, -1\}^n$ , the probability that  $\mathbf{x}$  has a compartment of sign  $\sigma$  is  $N(d, n)2^{-n}$ . In particular, the probability that there is a non-empty compartment of sign  $(1, 1, \dots, 1)$  is  $N(d, n)2^{-n}$ . Equivalently, the probability that some hemisphere contains all the points  $x_1, \dots, x_n$  is

$$p_{d,n} = N(d, n)2^{-n} = 2^{-n+1} \sum_{k=0}^{d-1} \binom{n-1}{k}.$$

□

**Notes.** The beautiful result above is a theorem of Wendel from 1962. In fact, the origin of the problem goes back to R.E. Machol who, in the early 1960s, challenged L.J. Savage to evaluate  $p_{3,4}$ . Savage did show that  $p_{3,4} = \frac{7}{8}$  and, more generally, that

$$p_{n,n+1} = 1 - 2^{-n}.$$

Savage then posed the problem of computing  $p(d, n)$  for all  $d$  and  $n$  to J.G. Wendel who, after a remark of D.A. Darling, conjectured the ‘duality’ relation

$$p_{m,m+n} + p_{n,m+n} = 1.$$

Wendel proved that this is indeed the case and gave the solution presented above.

J.G. Wendel, A problem in geometric probability, *Math. Scand.* **11** (1962), 109–111.

### 137. Planar and Spherical Triangles

If a spherical triangle is reproduced in the plane in such a way that the side-lengths are preserved then each angle of the planar triangle is smaller than the corresponding angle of the spherical triangle.

*Proof.* Let  $ABC$  be a spherical triangle and  $A'B'C'$  the corresponding planar triangle with  $AB = A'B' = c$ ,  $BC = B'C' = a$  and  $CA = C'A' = b$ . Let  $\alpha$  be the angle of  $ABC$  at  $A$ , and  $\alpha'$  the corresponding planar angle. Our task is to show that  $\alpha > \alpha'$ . Since  $0 < \alpha, \alpha' < \pi$ , this is equivalent to showing that  $\sin \frac{\alpha}{2} > \sin \frac{\alpha'}{2}$ . Note that with  $s = (a + b + c)/2$  we have

$$\sin \frac{\alpha}{2} = \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin b \sin c}}$$

and

$$\sin \frac{\alpha'}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Hence, our task is to show that

$$\frac{\sin(s-b) \sin(s-c)}{\sin b \sin c} > \frac{(s-b)(s-c)}{bc},$$

i.e.,

$$\frac{\sin(s-b)}{s-b} \frac{\sin(s-c)}{s-c} > \frac{\sin c}{c} \frac{\sin b}{b}. \quad (58)$$

To see that (58) does hold, simply note that the function  $\sin x/x$  is monotone decreasing for  $0 < x < \pi$ ,  $s-b = (a+c-b)/2 < ((b+c)+c-b)/2 = c$  and, equivalently,  $s-c < b$ .  $\square$

**Notes.** The result in the question is due to L.M. Blumenthal, the solution above was given by J.M. Kelly (Solution to Problem E 741, proposed by W. Scott, *Amer. Math. Monthly* **54** (1947), 283.).

### 138. Hobnails: Hadžiivanov's Theorem

Let  $f$  be a continuous map from the  $n$ -dimensional hobnail  $H^n \subset \mathbb{R}^{n+1}$  into  $\mathbb{R}^n$ . Then there are points  $x, y \in H^n$  at distance 1 such that  $f(x) = f(y)$  and if both  $x$  and  $y$  are in  $B^n$ , the unit ball in  $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ , then the unit segment  $[x, y]$  contains the centre of  $B^n$ .

*Proof.* Let  $C^n \subset \mathbb{R}^{n+1}$  be the following  $n$ -dimensional cylinder:

$$C^n = B^n \cup (B^n + e_{n+1}) \cup (S^{n-1} \times [0, 1]) = B^n \cup T^n \cup U^n,$$

the unit vector  $e_{n+1} = (0, \dots, 0, 1)$  is the  $(n+1)$ st canonical basis vector of  $\mathbb{R}^{n+1}$ , and the sphere  $S^{n-1}$  is the boundary of  $B^n$  in  $\mathbb{R}^n$ , as in Figure 87. We call  $B^n$  the *bottom*,  $T^n$  the *top*, and  $U^n$  the *side* of the cylinder  $C^n$ . Note that there is a natural homeomorphism  $\varphi$  mapping the unit sphere  $S^n \subset \mathbb{R}^{n+1}$  into the cylinder  $C^n$  such that

$$\varphi(-x) = e_{n+1} - \varphi(x) \text{ for every } x \in S^n.$$

Now, define a map  $\psi : C^n \rightarrow H^n$  as follows.

- (1) If  $x \in B^n$  then  $\psi(x) = x$ .
- (2) If  $u = x + he_{n+1} \in U^n$  with  $x \in S^{n-1}$ , then  $\psi(u) = (1 - h)x \in B^n$ .
- (3) If  $y = x + e_{n+1} \in T^n$ , so that  $x \in B^n$ , then  $\psi(y) = (1 - \|x\|^n)^{1/2}e_{n+1}$ .

Note that if  $z \in S^n$  then  $x = \psi(\varphi(z)) \in H^n$  and  $y = \psi(\varphi(-z)) \in H^n$  are at distance 1; furthermore, if  $x, y \in B^n$  then the unit interval  $[x, y]$  contains the centre of  $B^n$ .

After this preparation, the assertion is easily proved. Indeed, let  $f : H^n \rightarrow \mathbb{R}^n$  be continuous. Then so is the map  $g = f \circ \psi \circ \varphi : S^n \rightarrow \mathbb{R}^n$ . By Borsuk's theorem, in the form of assertion (2) of Problem 41, there is a point  $z \in S^n$  such that  $g(z) = g(-z)$ . Hence, for  $x = \psi(\varphi(z))$  and  $y = \psi(\varphi(-z))$  we have  $f(x) = f(y)$ , and the points  $x, y \in H^n$  are as required.  $\square$

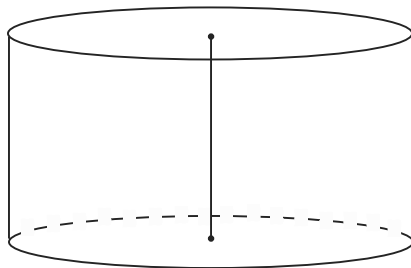


Fig. 87. The 2-dimensional cylinder  $C^2$ , with the hobnail  $H^2$  at its bottom and centre.

**Notes.** This result is due to N. Hadžiivanov, who introduced hobnails in order to prove this result.

N. Hadžiivanov, Continuous mappings of knobs into  $\mathbb{R}^n$  (in Russian), *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **22** (1974), 283–287.

N.G. Hadžiivanov, Continuous mappings of hobnails into Euclidean spaces (in Bulgarian), in *Mathematics and Mathematical Education* (Proc. Second Spring Conf. Bulgarian Math. Soc., Vidin, 1973), Izdat. B'lgar. Akad. Nauk., Sofia, (1974), pp. 221–230.



### 139. A Probabilistic Inequality

Let  $X$  be a random variable with probability density function  $f : \mathbb{R} \rightarrow [0, \infty)$ . Then, for  $p \geq 1$ , we have

$$\|f\|_{\infty} \|X\|_p \geq \frac{1}{2}(p+1)^{-1/p}.$$

Furthermore, equality holds if and only if  $X$  is uniformly distributed on some interval  $[-t, t]$ .

*Proof.* We may and shall assume that  $X$  is symmetric, i.e.,  $f$  is even:  $f(-t) = f(t)$ , and  $f$  is monotone increasing up to 0 and monotone decreasing after that. For  $x \geq 0$ , set

$$F(x) = \int_0^x f(t)dx$$

so that, trivially,  $F(0) = 0$ ,  $F(\infty) = 1/2$  and  $F(x) \leq x\|f\|_{\infty}$ . Hence,

$$\begin{aligned} 2^{-p} &= 2(F(\infty))^{p+1} = 2 \int_0^{\infty} (F(x)^{p+1})' dx \\ &= 2(p+1) \int_0^{\infty} f(x)F(x)^p dx \leq 2(p+1) \int_0^{\infty} f(x)(x\|f\|_{\infty})^p dx \\ &= (p+1)\|f\|_{\infty}^p \cdot 2 \int_0^{\infty} x^p f(x) dx = (p+1)\|f\|_{\infty}^p \|X\|_p^p. \end{aligned}$$

Equality holds if and only if  $f(t) = \|f\|$  or 0 for every  $t$ . □

**Notes.** See Problem 140.

## 140. Cube Slicing

For  $n \geq 2$ , every section of the solid unit cube  $I^n = \{\mathbf{x} = (x_i)_1^n : -1/2 \leq x_i \leq 1/2 \text{ for every } i\} \subset \mathbb{R}^n$  by an  $(n-1)$ -dimensional subspace  $H$  of  $\mathbb{R}^n$  has volume at least 1. Furthermore, the volume is 1 if and only if  $H$  is parallel to a face of the cube.

*Proof.* Let  $n \geq 2$ , and let  $H$  be an  $(n-1)$ -dimensional subspace with unit normal vector  $\mathbf{a} = (a_1, a_2, \dots, a_n)$ , with  $|a_i| < 1$  for every  $i$  (i.e.,  $a_i \neq 0$  for every  $i$ ). It suffices to show that  $|H \cap I^n|$ , the  $(n-1)$ -dimensional volume of the intersection of  $I^n$  with  $H$ , is less than 1.

Let  $X_1, X_2, \dots, X_n$  be independent random variables, each uniformly distributed in  $[1/2, 1/2]$ . Then the random vector  $(X_1, X_2, \dots, X_n)$  induces the standard measure on the cube  $I^n$ . Let us write  $f(r)$  for  $|(H + r\mathbf{a}) \cap I^n|$ , the  $(n-1)$ -dimensional volume of the intersection of  $I^n$  with a hyperplane parallel to  $H$ , at distance  $|r|$  from  $H$ . Since  $f$  is continuous,

$$\begin{aligned} f(r) &= \lim_{s \rightarrow r} \frac{1}{2s} \int_{r-s}^{r+s} f(t) dt \\ &= \lim_{s \rightarrow r} \frac{1}{2s} \left| \left\{ \mathbf{x} \in I^n : \left| \sum_{i=1}^n a_i x_i - r \right| \leq s \right\} \right| \\ &= \lim_{s \rightarrow r} \frac{1}{2s} \mathbb{P} \left( \left| \sum_{i=1}^n a_i X_i - r \right| \leq s \right). \end{aligned}$$

This shows that  $f(r)$  is the continuous probability density function of  $X = \sum_{i=1}^n a_i X_i$ .

Let us note an important property of  $f(r)$ : it is not only an even function, but attains its maximum at  $r = 0$ . This is an easy consequence of the Brunn–Minkowski inequality, stating that if  $A$  and  $B$  are convex sets in  $\mathbb{R}^k$  then

$$|(A + B)/2|^{1/k} \geq (|A|^{1/k} + |B|^{1/k})/2,$$

where, this time,  $|\cdot|$  stands for the  $k$ -dimensional volume. Applying this inequality to the slices of  $I^n$  giving  $f(r)$  and  $f(-r)$ , we find that

$$f(0)^{1/(n-1)} \geq (f(r)^{1/(n-1)} + f(-r)^{1/(n-1)})/2 = f(r)^{1/(n-1)},$$

and so  $f(0) \geq f(r)$ , as claimed.

After this preparation, let us see why  $f(0) > 1$ . Applying the inequality in Problem 139 to the random variable  $X = \sum_{i=1}^n a_i X_i$  and density function  $f$  with  $p = 2$ , and noting that  $f$  is not constant on an interval about 0, we find that

$$f(0) \|X\|_2 > \frac{1}{2} 3^{-1/2}.$$

Since  $\mathbb{E}(X^2) = (\sum_{i=1}^n a_i^2) \mathbb{E}(X_1^2) = 1/12$ , we see that  $\|X\|_2 = 1/2\sqrt{3}$ , and so  $f(0) > 1$ , as required.  $\square$

**Notes.** The beautiful result in this question is due to Hensley, who was also the first to use random variables to express the volume of a cube slice. The elegant and simple proof we have presented was given by Ball, who also proved the probabilistic inequality in Problem 139. Ball also proved the companion (and considerably deeper) result that no  $(n - 1)$ -dimensional slice of the  $n$ -dimensional unit cube has volume greater than  $\sqrt{2}$ . This result is trivially best possible, as one can see by taking a slice parallel to  $n - 2$  of the basis vectors and going through the diagonal of the face of the remaining two vectors.

K. Ball, Cube slicing in  $\mathbb{R}^n$ , *Proc. Amer. Math. Soc.* **97** (1986), 465–473.

D. Hensley, Slicing the cube in  $\mathbb{R}^n$  and probability, *Proc. Amer. Math. Soc.* **73** (1979), 343–353.

## 141. Measures on $[0, 1]$ : the Hobby–Rice Theorem

Let  $\mu_1, \mu_2, \dots, \mu_n$  be nonatomic signed Borel measures on  $[0, 1]$ . Then, for some  $r$ ,  $1 \leq r \leq n$ , there are points  $\xi_0 = 0 < \xi_1 < \dots < \xi_r < \xi_{r+1} = 1$ , such that with  $A = \bigcup_{i=0}^{\lfloor r/2 \rfloor} [\xi_{2i}, \xi_{2i+1}]$  and  $B = \bigcup_{i=1}^{\lfloor r/2 \rfloor} [\xi_{2i-1}, \xi_{2i}] = [0, 1] \setminus A$ , we have

$$\mu_j(A) = \mu_j(B) \text{ for every } j = 1, \dots, n.$$

*Proof.* Clearly, what we have to prove is that there are real numbers  $\rho_0 = 0 \leq \rho_1 \leq \dots \leq \rho_n \leq \rho_{n+1} = 1$  and signs  $\varepsilon_1, \dots, \varepsilon_{n+1}$ ,  $\varepsilon_i = \pm 1$ , such that

$$\sum_{i=1}^{n+1} \varepsilon_i \mu_j([\rho_{i-1}, \rho_i]) = 0 \quad (59)$$

for every  $j$ ,  $j = 1, \dots, n$ . To find the division points  $\xi_i$  (and so the sets  $A$  and  $B$ ) in the statement, simply unite those neighbouring intervals  $[\rho_{h-1}, \rho_h]$  that have the same sign.

Define a map  $f = (f_1, \dots, f_n) : S^n \rightarrow \mathbb{R}^n$  by setting, for  $x = (x_1, \dots, x_{n+1}) \in S^n$  and  $j = 1, \dots, n$ ,

$$f_j(x) = \sum_{i=1}^{n+1} (\text{sign } x_i) \mu_j([y_{i-1}, y_i]),$$

where  $y_i = \sum_{h=1}^i x_h^2$ , so that  $[y_{i-1}, y_i]$  has length  $x_i^2$ , and  $y_{n+1} = \sum_{h=1}^{n+1} x_h^2 = 1$ . Then  $f$  is a continuous antipodal map:  $f(-x) = -f(x)$  for every  $x \in S^n$ . Therefore, by Borsuk's theorem in the form of assertion (3) in Problem 41, we have  $f(x) = 0$  for some  $x \in S^n$ . Hence (59) holds with  $\rho_i = \sum_{h=1}^i x_h^2$  and  $\varepsilon_i = \text{sign } x_i$ .  $\square$

**Notes.** The Hobby–Rice theorem is a basic theorem in  $L_1$  approximation. The beautiful proof we have presented was found by Pinkus; a decade later, it was rediscovered by Alon and West.

N. Alon and D.B. West, The Borsuk–Ulam theorem and bisection of necklaces, *Proc. Amer. Math. Soc.* **98** (1986), 623–628.

C.R. Hobby and J.R. Rice, A moment problem in  $L_1$  approximation, *Proc. Amer. Math. Soc.* **16** (1965), 665–670.

C.A. Micchelli, Best  $L_1$  approximation by weak Chebyshev systems and the uniqueness of interpolating perfect splines, *J. Approximation Theory* **19** (1977), 1–14.

R.R. Phelps, Uniqueness of Hahn–Banach extensions and unique best approximation *Trans. Amer. Math. Soc.* **95** (1960), 238–255.

A. Pinkus, A simple proof of the Hobby–Rice theorem, *Proc. Amer. Math. Soc.* **60** (1976), 82–84.

D. Wulbert, Liapanov's and related theorems, *Proc. Amer. Math. Soc.* **108** (1990), 955–960.



## 142. Cutting a Necklace

Let  $c_1, c_2, \dots, c_n$  be a ‘necklace’, a sequence of  $k$  colours,  $1, \dots, k$ , say, in which each colour occurs an even number of times; say,  $i$  occurs  $2a_i$  times. Then there are  $r \leq n$  cuts that partition the sequence into  $r + 1$  blocks such that each  $i$  occurs  $a_i$  times in the odd numbered blocks (and so the same number of times in the even blocks). More formally, there are integers  $\ell_0 = 0 < \ell_1 < \dots < \ell_r < \ell_{r+1} = n$ ,  $r \leq n$ , such that, for each  $i$ , there are  $a_i$  indices  $j$  with  $c_j = i$  and  $\ell_{2h} < j \leq \ell_{2h+1}$ ,  $h = 0, 1, \dots$

Furthermore, fewer than  $n$  cuts may not suffice.

*Proof.* Note first that if the sequence starts with all the 1s (all  $2a_i$  of them), continues with all the 2s, and so on, ending with all the  $k$ s, then a ‘judicious’ partition needs  $k$  cuts: each run of  $2a_i \geq 2$  terms equal to  $i$  must be ‘cut’.

The main assertion that  $k$  cuts suffice is immediate from the Hobby–Rice theorem from Problem 141. Although this will be obvious to most readers, we shall take our time to set the scene. Given a necklace, i.e., a sequence  $c = (c_j)_{j=1}^n$  of our  $k$  in which each term is one of our  $k$  ‘colours’  $1, 2, \dots, k$ , for  $i = 1, \dots, k$ , define a function  $g_i$  on  $[0, n]$  by setting

$$g_i(t) = \begin{cases} 1 & \text{if } c_{\lceil t \rceil} = i, \\ 0 & \text{otherwise.} \end{cases}$$

Putting it slightly differently, if the  $j$ th bead has colour  $i$ , i.e.,  $c_j = i$  then the  $i$ th function  $g_i(t)$  is 1 for  $j - 1 \leq t \leq j$ . Furthermore, let  $\mu_i$  be the Borel measure with density function  $g_i(t)$ , so that  $\mu([x, y]) = \int_x^y g_i(t) dt$ .

By the Hobby–Rice theorem in Problem 141, with no more than  $k$  cuts we can partition  $[0, n]$  into two parts of equal  $\mu_i$ -measure for each  $i$ . Such a cut can be pulled back to a judicious cut of our necklace provided each cut is made at an integer point. To complete our proof, all we have to note is that if there is a judicious cut, then there is an integral judicious cut as well. Indeed, let us take a judicious cut in which the number of fractional cuts is minimal. If not every cut is an integer then some cut  $\xi_h$  cuts a bead  $c_j$ , i.e., if  $j - 1 < \xi_h < j$  for some  $j$ , then another bead  $c_{j'}$  of the same colour has to be cut as well. But then we can move both cuts forward, at the same speed, until one of them becomes an integer, and thereby reduce the number of fractional cuts.  $\square$

**Notes.** The result in this problem was proved by Goldberg and West in a considerably more complicated way. A little later, Alon and West showed that it follows

easily from Borsuk's theorem and, better still, it can be considered to be a special case of the Hobby–Rice theorem.

As a generalization of the problem considered here, it is natural to ask what happens if we have  $m$  thieves, rather than only two and, for each  $i$ ,  $i = 1, \dots, k$ , the number of beads of colour  $i$  is divisible by  $m$ . Clearly, if the colours occur contiguously then we need  $(m - 1)k$  cuts for a judicious division of the loot. Alon proved that this is many cuts suffice for all necklaces.

N. Alon, Splitting necklaces, *Advances in Mathematics* **63** (1987), 247–253.

N. Alon and D.B. West, The Borsuk–Ulam theorem and bisection of necklaces, *Proc. Amer. Math. Soc.* **98** (1986), 623–628.

C.H. Goldberg and D.B. West, Bisection of circle colourings, *SIAM J. Algebraic and Discrete Methods* **6** (1985), 93–106.

### 143. The Norm of an Operator: the Riesz–Thorin Interpolation Theorem

Let

$$L(x, y) = \sum_{i=1}^m \sum_{j=1}^n a_{ij} x_i y_j$$

be the complex bilinear form defined by an  $m$  by  $n$  complex matrix  $A = (a_{ij})$ . For  $\alpha > 0$ ,  $\beta > 0$ , let  $M_{\alpha, \beta}$  be the maximum of  $|L(x, y)|$  under the conditions  $\sum_{i=1}^m |x_i|^{1/\alpha} \leq 1$  and  $\sum_{j=1}^n |y_j|^{1/\beta} \leq 1$ . Then  $\log M_{\alpha, \beta}$  is a convex function of  $\alpha$  and  $\beta$ .

*Proof.* Although the Riesz–Thorin theorem is a fundamental result in operator theory, our main reason for including it in this collection of problems is its amazing proof found by Thorin some years after Marcel Riesz had proved the result. Thorin’s proof is an absolute gem: it is hard not to agree with Littlewood, who in his *Miscellany* called it the *most impudent and brilliantly successful idea in mathematics*. To paraphrase Littlewood, the proof is based on the very simple principles that:

- (a) in finding the supremum of a function subject to a number of independent conditions on the variables, we may take the conditions in any order;
- (b) the limit of convex functions is convex;
- (c) the supremum of an arbitrary family of convex functions is convex.

Consequently, if we can express  $\log M_{\alpha, \beta}$  as

$$\sup(\sup(\cdots(\sup |L(x, y)|) \cdots)),$$

with the innermost supremum (taken with all variables for the outer suprema fixed) convex, then we are done. *Thorin, however, takes his innermost supremum with respect to a variable that is not there!*

We start by proving the following easy consequence of Hadamard’s three-circles theorem.

**Claim.** Let  $f(s) = \sum_{k=1}^n a_k e^{b_k s}$ , where the  $a_k$  are complex numbers and the  $b_k$  reals, and, as usual in number theory,  $s = \sigma + it$ , with  $\sigma$  and  $t$  real numbers. Set  $m(\sigma) = \max |f(s)|$ . Then  $\log m(\sigma)$  is a convex function of  $\sigma$ .

To prove this claim, note that we may assume that the  $b_k$  are rationals, so that  $b_k = c_k/d$  for some integers  $c_1, \dots, c_n$  and  $d$ . Then for  $z = e^{s/d}$  we have



$f(s) = g(z)$ , where  $g$  is a polynomial in  $z$ . Furthermore,  $r = |z|$  is precisely  $e^{\sigma/d}$ , so the assertion follows by appealing to Hadamard's three-circles theorem which tells us that  $\log M(r)$  is a convex function of  $\log r$ , where  $M(r) = \max |g(z)|$ .

Turning to the proof of the theorem, let  $\alpha_0$ ,  $\alpha_1 = \alpha_0 + \lambda$ ,  $\beta_0$  and  $\beta_1 = \beta_0 + \mu$  be positive real numbers, and for  $0 \leq \sigma \leq 1$  set  $\alpha_\sigma = \alpha_0 + \lambda\sigma$  and  $\beta_\sigma = \beta_0 + \mu\sigma$ . Our task is to prove that  $\log M_{\alpha_\sigma, \beta_\sigma}$  is a convex function of  $\sigma$  for  $0 \leq \sigma \leq 1$ .

Writing

$$x_i = \xi_i^{\alpha_\sigma} e^{i\phi_i}, \quad y_j = \eta_j^{\beta_\sigma} e^{i\psi_j},$$

with  $\xi_i, \eta_j \geq 0$  and  $\phi_i, \psi_j$  real, we find that

$$M_{\alpha_\sigma, \beta_\sigma} = \sup_{(\phi, \psi, \xi, \eta)} \left| \sum_{i=1}^m \sum_{j=1}^n a_{ij} \xi_i^{\alpha_0 + \lambda\sigma} \eta_j^{\beta_0 + \mu\sigma} e^{i(\phi_i + \psi_j)} \right|, \quad (60)$$

with the supremum (really, maximum) subject to

$$\sum_{i=1}^m \xi_i \leq 1, \quad \sum_{j=1}^n \eta_j \leq 1 \quad \text{and} \quad \xi_i, \eta_j \geq 0.$$

Now, if in (60) we replace the real number  $\sigma$  by the complex number  $s = \sigma + it$  (for any  $t$ ) then the supremum is unaltered, since the best choice of  $\phi_i + \psi_j$  simply gets translated. But when taking the supremum with respect to  $(\phi, \psi, \xi, \eta, t)$ , we may take the variables in any order: in particular, we may start by taking the supremum with respect to the dummy variable  $t$  we have just introduced, so that

$$\log M_{\alpha_\sigma, \beta_\sigma} = \sup_{(\phi, \psi, \xi, \eta)} \log m(\sigma; \phi, \psi, \xi, \eta),$$

where

$$m(\sigma) = \sup_t |f(s)|,$$

with

$$\begin{aligned} f(s) &= f(s; \phi, \psi, \xi, \eta) = \sum_{i=1}^m \sum_{j=1}^n a_{ij} \xi_i^{\alpha_0 + \lambda s} \eta_j^{\beta_0 + \mu s} e^{i(\phi_i + \psi_j)} \\ &= \sum_{i=1}^m \sum_{j=1}^n a_{ij} \xi_i^{\alpha_0} \eta_j^{\beta_0} e^{i(\phi_i + \psi_j)} e^{b_{ij}s}, \end{aligned}$$

where the  $b_{ij}$  are real.

By the Claim,  $\log m(\sigma)$  is convex for all  $\sigma$ ; in particular, it is convex for  $0 \leq \sigma \leq 1$ . Consequently,  $\log M_{\alpha, \beta}$  is convex since it is the supremum of convex functions.  $\square$

**Notes.** To make the proof self-contained, we include here a proof of Hadamard's three-circles theorem, although it is (at least should be) in every standard course on complex functions. We follow the presentation in Titchmarsh, p. 172.

Let  $f(z)$  be an analytic function which is regular for  $r_1 \leq |z| \leq r_3$ . Let  $r_1 < r_2 < r_3$ , and let  $M_1, M_2$  and  $M_3$  be the maxima of  $|f(z)|$  on the three circles  $|z| = r_1, r_2$ , and  $r_3$ , respectively. Then

$$M_2^{\log(r_3/r_1)} \leq M_1^{\log(r_3/r_2)} M_3^{\log(r_2/r_1)}. \quad (61)$$

Writing  $\ell_i$  for the real logarithm of  $r_i$ , so that  $r_i = e^{\ell_i}$ , then, taking logarithms and dividing by  $\ell_3 - \ell_1$ , (61) becomes

$$\log M_2 \leq \frac{\ell_3 - \ell_2}{\ell_3 - \ell_1} \log M_1 + \frac{\ell_2 - \ell_1}{\ell_3 - \ell_1} \log M_3, \quad (62)$$

i.e., the three-circles theorem claims that  $\log M(r)$  is a convex function of  $\log r$ , where  $M(r) = \max_{|z|=r} |f(z)|$ .

To prove (62), set

$$\lambda = \frac{\log M_1 - \log M_3}{\ell_3 - \ell_1},$$

so that

$$e^{\lambda \ell_1} M_1 = r_1^\lambda M_1 = r_3^\lambda M_3 = e^{\lambda \ell_3} M_3,$$

and set  $F(z) = z^\lambda f(z)$ . Then  $F(z)$  is regular in the annulus  $r_1 \leq |z| \leq r_3$ , and  $|F(z)|$  is single-valued. Consequently, the maximum of  $|F(z)|$  is attained on the boundary of this annulus; in particular,

$$e^{\lambda \ell_2} M_2 = r_2^\lambda M_2 \leq \max\{r_1^\lambda M_1, r_3^\lambda M_3\} = r_1^\lambda M_1 = e^{\lambda \ell_1} M_1.$$

Taking logarithms, we find that

$$\begin{aligned} \log M_2 &\leq \lambda \ell_1 + \log M_1 - \lambda \ell_2 = \frac{\log M_1 - \log M_3}{\ell_3 - \ell_1} (\ell_1 - \ell_2) + \log M_1 \\ &= \frac{\ell_3 - \ell_2}{\ell_3 - \ell_1} \log M_1 + \frac{\ell_2 - \ell_1}{\ell_3 - \ell_1} \log M_3, \end{aligned}$$

proving (62), and so Hadamard's three-circles theorem.

B. Bollobás (ed), *Littlewood's Miscellany*, Cambridge University Press, (1986), iii + 200 pp.  
M. Riesz, Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires, *Acta Math.* **49** (1926), 465–497.

R. Salem and A. Zygmund, A convexity theorem, *Proc. Nat. Acad. Sci. USA* **34** (1948), 443–447.

G.O. Thorin, Convexity theorems generalizing those of M. Riesz and Hadamard with some applications, *Communications du Séminaire Mathématique de l'Université de Lund* [Medd. Lunds Univ. Mat. Sem.] **9** (1948), 1–58.

E.C. Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford University Press, (1939), x + 454 pp.

## 144. Uniform Covers

*Every finite set has only finitely many irreducible uniform covers.*

*Proof.* Suppose for a contradiction that an  $n$ -element set  $S$  has infinitely many irreducible uniform covers. Let  $A_1, A_2, \dots, A_{2^n}$  be an enumeration of the subsets of  $S$ , and then select infinite sequences  $\Gamma_i = (\mathcal{C}_i^{(j)})_{j=1}^\infty$ ,  $1 \leq i \leq 2^n$ , so that  $\Gamma_i$  is a subsequence of  $\Gamma_{i-1}$  and the number of copies of  $A_i$  in  $\mathcal{C}_i^{(j)}$  is a non-decreasing function of  $j$ . Since every infinite sequence of non-negative integers has an infinite non-decreasing subsequence, this can be done. Clearly, the covers in the final sequence  $\Gamma_{2^n}$  are nested. Hence if  $\mathcal{C}'$  and  $\mathcal{C}$  are any two distinct terms of  $\Gamma_{2^n}$ , with  $\mathcal{C}$  following  $\mathcal{C}'$ , then  $\mathcal{C}'$  is properly contained in  $\mathcal{C}$ , so  $\mathcal{C}$  is not irreducible. This contradiction completes our proof.  $\square$

**Notes.** Writing  $D(n)$  for the number of irreducible covers of  $[n]$ , Huckeman, Jurkat and Shapley proved that  $D(n) \leq (n+1)^{(n+1)/2}$  for all  $n$  (see the paper of Graver below). Related results have been proved by Alon and Berman and Füredi. The result in this weak form and the proof above are from Bollobás and Thomason.

N. Alon and K.A. Berman, Regular hypergraphs, Gordon's lemma and invariant theory, *J. Combin. Theory Ser. A* **43** (1986), 91–97.

B. Bollobás and A.G. Thomason, Projections of bodies and hereditary properties of hypergraphs, *Bull. London Math. Soc.* **27** (1995), 417–424.

Z. Füredi, Indecomposable regular graphs and hypergraphs, *Discrete Math.* **101** (1992), 59–64.

J.E. Graver, A survey of the maximum depth problem for indecomposable exact covers, in *Infinite and Finite Sets* (Keszthely, 1973), Proc. Colloq. Math. Soc. J. Bolyai **10**, North-Holland, (1976), pp. 731–743.

## 145. Projections of Bodies

If  $K$  is a body in  $\mathbb{R}^n$  and  $\mathcal{C}$  is a  $k$ -cover of  $[n]$  then

$$|K|^k \leq \prod_{A \in \mathcal{C}} |K_A|.$$

*Proof.* We prove the inequality by induction on  $n$ . The case  $n = 1$  being trivial, we turn to our proof of the induction step. For  $x \in \mathbb{R}$ , let  $K(x)$  be the slice of  $K$  at height  $x_n = x$ , i.e., let  $K(x)$  consist of the points of  $K$  with  $n$ th coordinate equal to  $x$ , so that  $|K| = \int |K(x)| dx$ , where  $|K(x)|$  is the  $(n - 1)$ -dimensional volume of  $K(x)$ . (Strictly speaking,  $K(x)$  is the empty set unless the slice is an  $(n - 1)$ -dimensional body, but this slight distinction makes no difference to the argument below.)

Let us split  $\mathcal{C}$  according to the containment of  $n$ : set  $\mathcal{C}' = \{A \in \mathcal{C} : n \in A\}$  and  $\mathcal{C}'' = \mathcal{C} \setminus \mathcal{C}'$ . Then  $|\mathcal{C}'| = k$  and  $\{A \setminus \{n\} : A \in \mathcal{C}'\} \cup \mathcal{C}''$  forms a  $k$ -cover of  $[n - 1]$ . Consequently, by the induction hypothesis,

$$\prod_{A \in \mathcal{C}'} |K(x)_{A \setminus \{n\}}| \prod_{A \in \mathcal{C}''} |K(x)_A| \geq |K(x)|^k.$$

Moreover, we have

$$\begin{aligned} |K_A| &= \int |K(x)_{A \setminus \{n\}}| dx && \text{for } A \in \mathcal{C}', \\ |K_A| &\geq |K(x)_A| && \text{for } A \in \mathcal{C}''. \end{aligned}$$

So, appealing to Hölder's inequality, we obtain

$$\begin{aligned} |K| &= \int |K(x)| dx \leq \int \left[ \prod_{A \in \mathcal{C}'} |K(x)_{A \setminus \{n\}}| \prod_{A \in \mathcal{C}''} |K(x)_A| \right]^{1/k} dx \\ &\leq \left[ \prod_{A \in \mathcal{C}''} |K_A| \right]^{1/k} \int \prod_{A \in \mathcal{C}'} |K(x)_{A \setminus \{n\}}|^{1/k} dx \\ &\leq \left[ \prod_{A \in \mathcal{C}''} |K_A| \right]^{1/k} \prod_{A \in \mathcal{C}'} \left[ \int |K(x)_{A \setminus \{n\}}| dx \right]^{1/k} \\ &= \left[ \prod_{A \in \mathcal{C}} |K_A| \right]^{1/k}, \end{aligned}$$

as required. □

**Notes.** The inequality above, proved by Alekseev, and Bollobás and Thomason, is a considerable extension of the classical Loomis–Whitney inequality stating that  $\prod_{i=1}^n |K_{[n]\setminus\{i\}}| \geq |K|^{n-1}$  holds for any body  $K$ . (For this inequality, see also the books of Hadwiger (p. 162) and Burago and Zalgaller (p. 95); for a streamlined proof, see Allan.)

G.R. Allan, An inequality involving product measures, in *Radical Banach Algebras and Automatic Continuity*, J.M. Bachar *et al.* (eds), Lecture Notes in Mathematics **975**, Springer-Verlag (1981), pp. 277–279.

V.E. Alekseev, On the entropy values of hereditary classes of graphs, *Discrete Appl. Math.* **3** (1993), 191–199.

Yu.D. Burago and V.A. Zalgaller, *Geometric Inequalities*, Springer-Verlag (1988), xiv + 331 pp.

H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer-Verlag (1957), xiii + 312 pp.

L.H. Loomis and H. Whitney, An inequality related to the isoperimetric inequality, *Bull. Amer. Math. Soc.* **55** (1949), 961–962.

## 146. BTBT: the Box Theorem of Bollobás and Thomason

For every body  $K$  in  $\mathbb{R}^n$  there is a box  $B$  such that  $|B| = |K|$  and  $|B_A| \leq |K_A|$  for every  $A \subseteq [n]$ .

*Proof.* Let  $\Gamma$  be the set of irreducible uniform covers of  $[n]$ . We know from Problem 144 that  $\Gamma$  is a finite set. For a  $k$ -cover  $\mathcal{C} \in \Gamma$ , the inequality in Problem 145 tells us that

$$\prod_{A \in \mathcal{C}} |K_A| \geq |K|^k.$$

Furthermore,

$$\prod_{i \in S} |K_{\{i\}}| \geq |K_S|$$

for every  $S \subseteq [n]$ . (This inequality is a special case of the previous one, and is also immediate from first principles.)

In this way we have written down a finite set of inequalities involving the numbers  $\{|K_A| : A \subseteq [n]\}$ . Let  $\{x_A : A \subseteq [n]\}$  be a collection of positive numbers with  $x_A \leq |K_A|$  and  $x_{[n]} = |K|$ , which are minimal subject to satisfying all the above inequalities with  $x_A$  in place of  $|K_A|$  for all  $A$ . Note that we are applying only a finite number of constraints to the  $x_A$ , but that nevertheless  $\prod_{A \in \mathcal{C}} x_A \geq |K|^k$  for every  $k$ -cover  $\mathcal{C}$ , any such cover being a disjoint union of irreducible uniform covers.

The fact that the numbers  $x_{\{i\}}$ ,  $i \in [n]$ , are minimal means that for each  $i$  there is an inequality involving  $x_{\{i\}}$  in which equality holds. If the inequality is of the first kind, then we have a  $k_i$ -cover  $\mathcal{C}_i$  of  $[n]$  with  $\{i\} \in \mathcal{C}_i$  and  $\prod_{A \in \mathcal{C}_i} x_A = |K|^{k_i}$ . The same is in fact true if the inequality is of the second kind, namely  $\prod_{i \in S} x_{\{i\}} = x_S$  for some  $S \subseteq [n]$ , because in this case the minimality of  $x_S$  implies  $\prod_{A \in \mathcal{C}} x_A = |K|^{k_i}$  for some  $k_i$ -cover  $\mathcal{C}$  of  $[n]$  containing  $S$ , and we can take  $\mathcal{C}_i = (\mathcal{C} \setminus \{S\}) \cup \{\{i\} : i \in S\}$ . Now we let  $\mathcal{C} = \bigcup_{i=1}^n \mathcal{C}_i$  and  $k = \sum_{i=1}^n k_i$ . Then  $\mathcal{C}$  is a  $k$ -cover of  $[n]$  containing  $\{i\}$  for all  $i \in [n]$ , and  $\prod_{A \in \mathcal{C}} x_A = |K|^k$ . But  $\mathcal{C}' = \mathcal{C} \setminus \{\{i\} : i \in [n]\}$  is a  $(k-1)$ -cover of  $[n]$ , so  $\prod_{A \in \mathcal{C}'} x_A \geq |K|^{k-1}$ , from which we conclude  $\prod_{i=1}^n x_{\{i\}} \leq |K|$ . Since  $\{\{i\} : i \in [n]\}$  is a 1-cover of  $[n]$  the reverse inequality also holds here, and hence, in fact, equality holds.

Finally, we observe that, for any  $A \subseteq [n]$ , the set  $\{A\} \cup \{\{i\} : i \notin A\}$  is a 1-cover of  $[n]$ , so

$$|K| \leq x_A \prod_{i \notin A} x_{\{i\}} \leq \prod_{i \in A} x_{\{i\}} \prod_{i \notin A} x_{\{i\}} = |K|,$$

whence  $x_A = \prod_{i \in A} x_{\{i\}}$ . It follows that the box  $B$ , of side-length  $x_{\{i\}}$  in the direction of  $v_i$ , satisfies  $|B| = |K|$  and  $|B_A| = x_A \leq |K_A|$  for all  $A \subseteq [n]$ .  $\square$

B. Bollobás and A. Thomason, Projections of bodies and hereditary properties of hypergraphs, *Bull. London Math. Soc.* **27** (1995), 417–424.



## 147. Intersecting Uniform Set Systems: the Ray-Chaudhuri–Wilson Inequality

Let  $k \geq 1$  and  $S \subset [k-1]$ , with  $s = |S| \geq 1$ . Then a  $k$ -uniform, strictly  $S$ -intersecting family of subsets of  $[n] = \{1, \dots, n\}$  consists of at most  $\binom{n}{s}$  sets.

*Proof.* Let  $x_1, x_2, \dots, x_n$  be independent variables, and for  $I \subset [n]$  set  $x_I = \prod_{i \in I} x_i$ . Furthermore, let  $\mathcal{P}(n) = \mathcal{P}([n]) \cong Q_n = \{0, 1\}^n \subset \mathbb{R}^n$  be the power set of  $[n]$ . Every set  $I \in \mathcal{P}(n)$  is naturally identified with its characteristic function  $\chi_I$ . Thus, given a function  $g(x_1, \dots, x_n)$ , we write  $g(I)$  for the evaluation of  $g$  at  $x_i = 1$  if  $i \in I$  and  $x_i = 0$  otherwise.

**Claim 1.** If  $f : \mathcal{P}(n) \rightarrow \mathbb{R}$  is such that  $f(I) \neq 0$  whenever  $|I| \leq r$  then the set  $\{x_I f : |I| \leq r\}$  of functions on  $\mathcal{P}(n)$  is linearly independent.

To justify this claim, observe that

$$x_I(J) = \begin{cases} 1 & \text{if } I \subset J, \\ 0 & \text{otherwise.} \end{cases}$$

Now, suppose that

$$\sum_{I: |I| \leq r} \lambda_I x_I f = 0$$

is a non-trivial relation. Let  $I_0$  be such that  $\lambda_{I_0} \neq 0$  and  $|I_0|$  is minimal. Then

$$\sum_{I: |I| \leq r} \lambda_I x_I(I_0) f(I_0) = \lambda_{I_0} f(I_0) \neq 0,$$

and this contradiction shows that the set  $\{x_I f : |I| \leq r\}$  is linearly independent, as required.

Continuing our proof, as suggested in part (ii) of the *Hint*, let  $S = \{r_1, r_2, \dots, r_s\}$ ,  $\mathcal{F} = \{A_1, A_2, \dots, A_m\}$ , and define  $f_i : Q_n \rightarrow \mathbb{R}$  by

$$f_i(x) = \prod_{h=1}^s \left( \sum_{j \in A_i} x_j - r_h \right) \quad \text{for } x \in Q_n.$$

Note that  $f_i(A_j) = \prod_{h=1}^s (|A_i \cap A_j| - r_h) \neq 0$  if  $i = j$ , and  $f_i(A_j) = 0$  if  $i \neq j$ , so the functions  $f_1, f_2, \dots, f_m$  are linearly independent. We shall show that more is true: a considerably larger set is also independent. Set

$$F = \{f_1, f_2, \dots, f_m\} \cup \{x_I f : |I| \leq s-1\},$$

where  $f = \sum_{j=1}^n x_j - k$ .

**Claim 2.** *The set  $F$  is also independent.*

To prove this claim, assume for a contradiction that there is a linear relation

$$\sum_{i=1}^m \lambda_i f_i + \sum_{I: |I| \leq s-1} \mu_I x_I f = 0 \quad (63)$$

among the elements of  $F$ . Evaluating the left-hand-side at  $A_i$ , we find that all the terms of the second sum vanish, since  $|A_i| = k$ ; consequently, as  $f_j(A_i) = 0$  whenever  $j \neq i$ , the entire sum is  $\lambda_i f_i(A_i) = \lambda_i$ . Hence  $\lambda_i = 0$  for every  $i$ , and so (63) becomes

$$\sum_{I: |I| \leq s-1} \mu_I x_I f = 0.$$

Since the intersection of two  $k$ -sets has at most  $k-1$  elements and  $\mathcal{F}$  is strictly  $S$ -intersecting,  $s \leq k$ . Therefore,  $f(I) = |I| - k < 0$  for all  $I$  with  $|I| \leq s-1$ ; consequently, Claim 1 implies that  $\mu_I = 0$  for every  $I$  with  $|I| \leq s-1$ . This proves Claim 2.

Having proved these assertions, it is very easy to complete our proof of this result. Indeed, we know that  $F$  is a collection of  $m + \sum_{i=0}^{s-1} \binom{n}{i}$  linearly independent functions, and each of these functions is a multilinear polynomial of degree at most  $s$ , i.e., of the form  $\sum_{I: |I| \leq s} \lambda_I x_I$ . Since the dimension of the space of multilinear polynomials of degree at most  $s$  in  $n$  variables is  $\sum_{i=0}^s \binom{n}{i}$ , we find that  $m \leq \binom{n}{s}$ , completing our proof.  $\square$

**Notes.** Rather than assuming that  $\mathcal{F}$  is strictly  $S$ -intersecting, we could assume that  $\mathcal{F}$  is a  $k$ -uniform  $S$ -intersecting system and  $s = |S| \leq k$ . Without conditions like these, the conclusion is not correct since *every* set system on  $[n]$  is  $S$ -intersecting with  $S = \{0, 1, \dots, n-1\}$ , while  $\binom{n}{s} = 1$  with  $s = |S| = n$ .

The fundamental result in this problem is the *Ray-Chaudhuri–Wilson inequality*, whose non-uniform version, the *Frankl–Wilson inequality*, is Problem 148. The beautiful proof presented above was given by Alon, Babai and Suzuki; it was inspired by the methods of Larman, Rogers and Seidel, Babai and Blokhuis; see Problems 47 and 148.

N. Alon, L. Babai and H. Suzuki, Multilinear polynomials and Frankl–Ray–Chaudhuri–Wilson type intersection theorems, *J. Combin. Theory Ser. A* **58** (1991), 165–180.

A. Blokhuis, A new upper bound for the cardinality of 2-distance sets in Euclidean space, in *Convexity and Graph Theory* (Jerusalem, 1981), North-Holland Math. Stud., **87**, North-Holland, (1984), pp. 65–66.

D.G. Larman, C.A. Rogers and J.J. Seidel, On two-distance sets in Euclidean space, *Bull. London Math. Soc.* **9** (1977), 261–267.

D.K. Ray-Chaudhuri and R.M. Wilson, On  $t$ -designs, *Osaka J. Math.* (**12** (1975), 737–744.

## 148. Intersecting Set Systems: the Frankl–Wilson Inequality

For  $S \subset [n] = \{1, \dots, n\}$ , a collection of sets is said to be  $S$ -intersecting if  $|A \cap B| \in S$  whenever  $A$  and  $B$  are distinct members of it. Let  $\mathcal{F}$  be an  $S$ -intersecting family of subsets of  $[n]$ . Then

$$|\mathcal{F}| \leq \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{s}, \quad (64)$$

where  $s = |S|$ .

*Proof.* Let  $S = \{r_1, r_2, \dots, r_s\}$  and  $\mathcal{F} = \{A_1, A_2, \dots, A_m\}$ , where  $|A_1| \leq |A_2| \leq \dots \leq |A_m|$ . For  $1 \leq i \leq m$ , let  $a_i = \mathbf{1}_{A_i} \in \mathbb{R}^n$  be the characteristic vector of  $A_i$ ; thus  $a_i = (a_{i1}, a_{i2}, \dots, a_{in})$ , where  $a_{ij}$  is 1 if  $j \in A_i$  and 0 otherwise. Also, for  $x, y \in \mathbb{R}^n$  write  $\langle x, y \rangle$  for the inner product of the vectors  $x$  and  $y$ , so that  $\langle a_i, a_j \rangle = |A_i \cap A_j|$ .

For  $1 \leq i \leq m$ , set

$$f_i(x) = f_i(x_1, \dots, x_n) = \prod_{j: r_j < |A_i|} (\langle a_i, x \rangle - r_j), \quad (65)$$

so that  $f_i(x)$  is a polynomial in  $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$ . Since

$$f_i(a_j) = 0 \text{ for } 1 \leq j < i \leq m, \quad (66)$$

and

$$f_i(a_i) \neq 0 \text{ for } 1 \leq i \leq m, \quad (67)$$

the polynomials  $f_1, f_2, \dots, f_m$  are independent over  $\mathbb{R}$ .

Given a polynomial  $f(x) \in \mathbb{R}[x]$ , let  $\bar{f}(x)$  be the multilinear polynomial obtained from it by taking its expansion into monomials and for each monomial reducing the exponent of each variable that occurs in it to 1. Thus, the polynomial  $f(x) = 2x_1^3x_4x_7^{17} - x_2^2x_4^2x_7$  is mapped into  $\bar{f}(x) = 2x_1x_4x_7 - x_2x_4x_7$ , and  $g(x) = 3x_2^3x_5^2 - 2x_2^2x_5^6$  into  $\bar{g}(x) = x_2x_5$ . Note that if  $x$  is a 0-1 vector (i.e., all its coordinates are 0 or 1; or, putting it slightly differently is a point of the cube  $\{0, 1\}^n \subset \mathbb{R}^n$ ) then  $f(x) = \bar{f}(x)$ . In particular, the polynomials  $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_m$  also satisfy relations (66) and (67), and so are independent.

On the other hand, relation (65), the definition of  $f_i(x)$ , implies that each  $f_i$  has degree at most  $s$ , so each polynomial  $\bar{f}_i$  is a linear combination of monomials of the form  $x_{k_1}x_{k_2} \dots x_{k_t}$ , where  $0 \leq t \leq s$ . As there are  $\sum_{\ell=0}^s \binom{n}{\ell}$  such monomials,  $m$  is also at most that large, as claimed.  $\square$

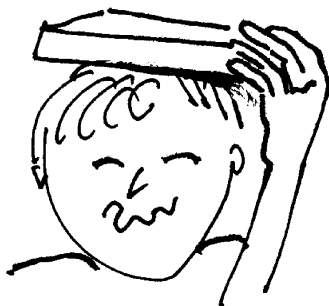
**Notes.** The Frankl–Wilson inequality we have just proved is also known as the *non-uniform Ray-Chaudhuri–Wilson inequality*. (The original Ray-Chaudhuri–Wilson inequality concerns uniform set systems, i.e., set systems in which all sets have the same size; see Problem 147.) The gem of a proof reproduced above is due to Babai; it was inspired by the method Larman, Rogers and Seidel applied to prove their result in Problem 47.

The bound in the result is clearly best possible, as shown by the set system  $\mathcal{F} = [n]^{(\leq s)}$  consisting of all subsets of  $[n]$  with at most  $s$  elements. Indeed, the cardinality of the intersection of two elements of this set system is  $0, 1, \dots, s - 2$  or  $s - 1$ .

L. Babai, A short proof of the nonuniform Ray-Chaudhuri–Wilson inequality, *Combinatorica* **8** (1988), 133–135.

P. Frankl and R.M. Wilson, Intersection theorems with geometric consequences, *Combinatorica* **1** (1981), 357–368.

D.K. Ray-Chaudhuri and R.M. Wilson, On  $t$ -designs, *Osaka J. Math.*, **12** (1975), 737–744.



## 149. Maps from $S^n$

Let  $0 < d \leq 2$ , and let  $f$  be a continuous mapping of the  $n$ -dimensional unit sphere  $S^n \subset \mathbb{R}^{n+1}$  into  $\mathbb{R}^n$ . Then there are points  $a, b \in S^n$  at Euclidean distance  $d(a, b) = d$  with  $f(a) = f(b)$ .

*Proof.* For  $d = 2$ , this is just part (2) of Borsuk's theorem in Problem 41, so we may and shall assume that  $0 < d < 2$ .

Let  $c$  be a point of the sphere  $S^n$  such that the  $n$ th coordinate of  $f(c)$  is maximal. We may assume that  $c$  is the north pole of  $S^n$  and  $c = 0$ , as in Figure 88. Thus the unit sphere  $S^n$  is not in its canonical position, but centred about  $(0, \dots, 0, -1)$ . We may also assume that  $f(c) = 0$ , so  $f(S^n)$  is 'below' the hyperplane  $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ . (Needless to say, these assumptions are made only to help us to 'see' what is going on.) Let  $C^n$  be the spherical cap in  $S^n$ , with centre  $c$  and radius  $d$ . Let  $\varphi$  be a homeomorphism mapping the unit ball  $B^n$  of  $\mathbb{R}^n \subset \mathbb{R}^{n+1}$  with centre  $c = 0$  into  $C^n$ , keeping  $c$  fixed, such that if  $x, y \in B^n$ , the closed interval  $[x, y]$  contains  $c$ , and  $d(x, y) = 1$  then  $d(\varphi(x), \varphi(y)) = d$ . Thus  $\varphi$  'bends' the ball  $B^n$  into  $C^n$ ; it is trivial that there is such a homeomorphism. Let  $H^n \subset \mathbb{R}^{n+1}$  be the hobnail with head  $B^n$  and tang  $I$  from  $c = (0, \dots, 0)$  to  $(0, \dots, 0, 1)$ .

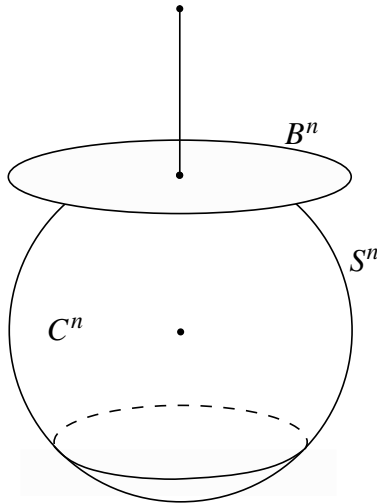


Fig. 88. The sphere  $S^n$ , the spherical cap  $C^n$ , and the ball  $B^n$  in  $\mathbb{R}^{n+1}$ .

So far, we have only defined a map and some sets, but have not given any arguments. Nevertheless, the proof is almost complete. Define  $g : H^n \rightarrow \mathbb{R}^n$  as follows:

- (1) for  $x = (0, \dots, 0, 0, h) \in I$ , set  $g(x) = (0, \dots, 0, h)$ ;
- (2) for  $x \in B^n$ , set  $g(x) = f(\phi(x))$ .

Thus  $g$  is a one-to-one mapping (in fact, isometry) from the tang of the hobnail into the straight-line segment in  $\mathbb{R}^n$  from  $(0, \dots, 0)$  to  $(0, \dots, 0, 1)$ , ‘pointing away’ from  $f(S^n)$ , sharing only the origin with  $g(B^n)$ .

By Hadžiivanov’s hobnail theorem in Problem 138, there are points  $x, y \in H^n$  at distance 1 with  $g(x) = g(y)$ . But then  $x, y \in B^n$  so, by the second assertion in Problem 138,  $0 \in [x, y]$ . Hence,  $d(\phi(x), \phi(y)) = d$ , so for  $a = \phi(x) \in S^n$  and  $b = \phi(y) \in S^n$  we have  $d(a, b) = d$  and  $f(a) = f(b)$ .  $\square$

**Notes.** This result is due to Hopf; its beautiful proof presented above was given by Nikiforov, and published by Hadžiivanov.

N. Hadžiivanov, On a theorem of Hopf (in Russian), *Annuaire Univ. Sofia Fac. Math. Inform.* **85** (1991), 65–69.

H. Hopf, Eine Verallgemeinerung bekannter Abbildungs- und Überdeckungssätze, *Portugaliae Math.* **4** (1944), 129–139.

### 150. Closed Covers of $S^n$ : Hopf's Theorem

Let  $0 < d \leq 2$ , and let  $F_1, \dots, F_{n+1}$  be closed sets covering the  $n$ -dimensional unit sphere  $S^n$ . Then there an index  $i$ ,  $1 \leq i \leq n+1$ , such that  $F_i$  contains two points at distance  $d$ .

*Proof.* We proceed as in the proof of the implication  $(2) \Rightarrow (4)$  in Problem 41. Thus, suppose that none of  $F_1, \dots, F_n$  contains two points at distance  $d$ . For  $1 \leq i \leq n$ , let

$$E_i = \{x : d(x, y) = d \text{ for some } y \in F_i\}.$$

Then each set  $E_i$  is closed and disjoint from  $F_i$ .

Let  $f_i(x) = d(x, E_i) - d(x, F_i)$ , where  $d(x, A)$  is the distance of  $x$  from a set  $A$ . Clearly, each function  $f_i$  is continuous, and  $f_i(x) > 0 > f_i(y)$  for  $x \in F_i$  and  $y \in E_i$ . Therefore  $f = (f_1, \dots, f_n)$  is a continuous map  $S^n \rightarrow \mathbb{R}^n$ . Hence, by the result in Problem 149,  $f(x) = f(y)$  for some points  $x$  and  $y$  at distance  $d$ . Since  $f_i(x) = f_i(y)$ , neither  $x$ , nor  $y$  belongs to  $F_i$ ,  $i = 1, \dots, n$ . Consequently,  $x$  and  $y$  are both in  $F_{n+1}$ .  $\square$

**Notes.** This result is Hopf's extension of the Lusternik–Schnirelmann–Borsuk theorem, assertion (4) in Problem 41.

H. Hopf, Eine Verallgemeinerung bekannter Abbildungs- und Überdeckungssätze, *Portugaliae Math.* **4** (1944), 129–139.



## 151. Spherical Pairs

Let  $X$  be a compact metric space and  $\varphi$  a distance-preserving involution on  $X$ . Then the pair  $(X, \varphi)$  is 1-spherical if and only if  $X$  contains a component  $C$  such that  $C = \varphi(C)$ . Also, if  $(X, \varphi)$  is not 1-spherical then  $X$  can be partitioned into disjoint compact sets  $W_1$  and  $W_2$  such that  $\varphi(W_1) = W_2$ .

*Proof.* (i) Suppose that  $X$  contains no component  $C$  such that  $C = \varphi(C)$ , and so for every component  $C$  we have  $C \cap \varphi(C) = \emptyset$ . Our aim is to construct a partition  $X = W \cup \varphi(W)$  where  $W$  is a clopen (closed and open) set. Then, in particular,  $X$  has no component  $C$  with  $\varphi(C) = C$ .

Given a component  $C$ , as  $C \cap \varphi(C) = \emptyset$ , there are disjoint clopen sets  $U_C$  and  $U_{\varphi(C)}$  such that  $C \subset U_C$ ,  $\varphi(C) \subset U_{\varphi(C)}$  and  $U_{\varphi(C)} = \varphi(U_C)$ . The family of open sets  $\{U_C : C \text{ is a component of } X\}$  covers  $X$ ; since  $X$  is compact, this cover has a finite subcover  $M_0 = \{U_{C_1}, \dots, U_{C_p}\}$ . Set  $M_1 = M_0 \cup \{U_{\varphi(C_1)}, \dots, U_{\varphi(C_p)}\}$  and let  $M$  be the family of all non-empty finite intersections of elements of  $M_1$ , i.e., the *atoms* of this system. Note that the members of  $M$  are disjoint clopen sets and if  $U \in M$ , then  $\varphi(U) \in M$  and  $U \cap \varphi(U) = \emptyset$ . Consequently,  $M = \{W_1, \dots, W_q, \varphi(W_1), \dots, \varphi(W_q)\}$ . Setting  $W = \bigcup_{i=1}^q W_i$ , we see that  $X = W \cup \varphi(W)$ .

From this it follows that  $(X, \varphi)$  is not a 1-spherical pair. Indeed, define  $f(W) = 1$ ,  $f(\varphi(W)) = -1$ . Then  $f$  is continuous and  $f(x) \neq f(\varphi(x))$  for every  $x \in X$ .

(ii) Suppose  $\varphi(C) = C$  for some component  $C$  of  $X$ . Let  $f : X \rightarrow \mathbb{R}$  be continuous. Define  $g : C \rightarrow \mathbb{R}$  by

$$g(x) = f(\varphi(x)) - f(x),$$

so that  $g(x) = -g(\varphi(x))$ . This implies that  $g(x) = 0$ , i.e.,  $f(x) = f(\varphi(x))$ , for some  $x \in C$ , since otherwise  $C = g^{-1}((-\infty, 0)) \cup g^{-1}((0, \infty))$ , contradicting the assumption that  $C$  is a component of  $X$ , in particular, is connected.  $\square$

## 152. Realizing Distances

Let  $(X, \varphi)$  be an  $n$ -spherical pair, as in Problem 151. Then

(i) for every closed cover  $X = \bigcup_{i=1}^{n+1} F_i$  and every  $0 < d \leq \text{diam}(X)$  there exists  $j \in [n+1]$  such that  $F_j$  realizes  $d$ ;

(ii) for every closed cover  $X = \bigcup_{i=1}^n F_i$ , there exists an index  $j \in [n]$  such that  $F_j$  realizes all distances  $0 < d \leq \text{diam}(X)$ .

*Proof.* We apply induction on  $n$ . Let  $n = 1$ . In this case the result in Problem 151 implies that  $X$  contains a component  $C$  such that  $C = \varphi(C)$ . Then (ii) is trivial since  $C \subset F_1$ . To prove (i), set  $G_1 = C \cap F_1$  and  $G_2 = C \cap F_2$ . Since  $G_1$  and  $G_2$  are closed and  $C = G_1 \cup G_2$  is connected, we have  $G_1 \cap G_2 \neq \emptyset$ . Select  $x \in G_1 \cap G_2$ . Observing that for every  $0 < d \leq \text{diam}(X)$  there exists  $y \in C$  with  $d(x, y) = d$ , the assertion follows.

As we shall see now, the induction steps for the two assertions can be proved simultaneously. If  $F_1$  realizes all distances, there is nothing to prove. Otherwise, set  $Y = F_1 \cup \varphi(F_1)$ , so that  $Y$  is invariant under the isometric involution  $\varphi$ , and note that  $\text{diam}(Y) = \text{diam}(X)$ . Since  $F_1$  does not realize all distances, there is a  $d$ ,  $0 < d \leq \text{diam}(X)$ , such that if  $x, y \in F_1$  then  $d(x, y) \neq d$ . By the case  $n = 1$  proved above,  $Y$  is not 1-spherical. Hence, by Problem 151,  $Y$  has a partition into compact sets  $W_1, W_2$  with  $\varphi(W_1) = W_2$ . Define a continuous odd map  $f : X \rightarrow \mathbb{R}$  by

$$f = \frac{d(x, W_1) - d(x, \varphi(W_2))}{d(x, W_1) + d(x, \varphi(W_2))},$$

so that  $f(\varphi(x)) = -f(x) \neq 0$  for every  $x \in Y$ . Set  $Z = f^{-1}(0)$ , and observe that  $Z = \varphi(Z)$  and  $\text{diam}(Z) = \text{diam}(X)$ .

We claim that  $Z$  is  $(n-1)$ -spherical. Indeed, otherwise there is a continuous map  $g : Z \rightarrow \mathbb{R}^{n-1}$  such that  $g(\varphi(z)) \neq g(z)$  for every  $z \in Z$ . By the Tietze–Urysohn extension theorem (see, e.g., p. 88 of Bollobás),  $g$  has an extension to a continuous function  $h$  defined on the whole space  $X$ , i.e., there is a continuous map  $h : X \rightarrow \mathbb{R}^{n-1}$  such that  $h(z) = g(z)$  whenever  $z \in Z$ . Then  $F = (f, h) : X \rightarrow \mathbb{R} \oplus \mathbb{R}^{n-1} = \mathbb{R}^n$  is continuous and  $F(\varphi(x)) \neq F(x)$  for every  $x \in X$ . This contradiction proves our claim.

Finally, if  $X = \bigcup_{i=1}^{n+1} F_i$  then  $Z \subset \bigcup_{i=2}^{n+1} F_i$ . Since  $Z$  is  $(n-1)$ -spherical, by the induction hypothesis for (i), for every distance there is a set  $F_i$  realizing it. Also, if  $X = \bigcup_{i=1}^n F_i$  then  $Z \subset \bigcup_{i=2}^n F_i$  so, by the induction hypothesis for (ii), some  $F_i$  realizes all the distances. □

**Notes.** Part (i) is an extension of the classical result of Hopf we encountered in Problem 150. Hadwiger conjectured that considerably more is true: the quantifiers can be interchanged. Thus, given a covering of  $S^n$  with  $n + 1$  closed sets  $F_1, \dots, F_{n+1}$ , not only is it true that for every distance  $0 < d \leq 2$  there is a set  $F_j$  realizing  $d$ , but, in fact, there is a set  $F_j$  that realizes *all* distances  $0 < d \leq 2$ . Hadwiger himself proved his conjecture for  $n = 1, 2$ ; Problem 3(ii) is precisely the case  $n = 1$ . However, the full conjecture is still open. To state some additional partial results, let  $h(n)$  be the maximal integer  $m$  such that for every covering of  $S^n$  with  $m$  closed sets  $F_1, \dots, F_m$  there is an index  $j$  such that  $F_j$  realizes all distances  $0 < d \leq 2$ . It is trivial that  $h(n) \leq n + 2$ , so Hadwiger's conjecture is precisely that  $h(n) = n + 1$ . Following the method of Lusternik and Schnirelmann and constructing certain vector fields on the sphere, Larman proved that  $h(n) \geq n/2$ .

Part (ii) above is a considerably stronger result which implies that  $h(n) \geq n$ , so  $n \leq h(n) \leq n + 1$ .

The two assertions in this problem are unpublished results of Nikiforov from 1978; essentially the same results were proved by Bogatyĭ in 1986. The proofs above closely follow that of Nikiforov.

B. Bollobás, *Linear Analysis*, Cambridge University Press, (1990), xi + 240 pp.

S.A. Bogatyĭ, Topological methods in combinatorial problems, (in Russian) *Uspekhi Mat. Nauk* **41** (1986), 37–48. English translation: *Russian Math. Surveys* **41** (1986), 43–57.

H. Hadwiger, Ein Überdeckungssatz für den Euklidischen Raum, *Portugaliae Math.* **4** (1944), 140–144.

H. Hopf, Eine Verallgemeinerung bekannter Abbildungs- und Überdeckungssätze, *Portugaliae Math.* **4** (1944), 129–139.

D.G. Larman, On the realization of distances within coverings of an  $n$ -sphere, *Mathematika* **14** (1967), 203–206.

### 153. A Closed Cover of $S^2$

Let  $f : S^2 \rightarrow S^2 \subset \mathbb{R}^3$  be a continuous map, and let  $S^2 = F_1 \cup F_2 \cup F_3$ , where the sets  $F_i$  are closed. Then for some  $x \in S^2$ , the points  $x$  and  $f(x)$  belong to the same set  $F_i$ .

*Proof.* Suppose for a contradiction that there is no point  $x \in S^2$  such that, for some  $i$ ,  $x, f(x) \in F_i$ . As the sets  $F_i$  cover  $S^2$ , this implies that  $f$  has no fixed point. Consequently, appealing to Brouwer's fixed point theorem (see, e.g., p. 216 of Bollobás), we find that  $f$  is a surjection.

Also,  $F_1 \cap F_2 \cap F_3 = \emptyset$ , since if  $x$  is in this intersection and  $f(x) \in F_i$ , then  $x$  and  $f(x)$  both belong to the same set  $F_i$ . For  $x \in S^2$ , let  $D_i(x)$  be the distance  $d(x, F_i)$  of  $x$  from  $F_i$ , and set  $D(x) = D_1(x) + D_2(x) + D_3(x)$ . Since no point is in all three  $F_i$ , this function  $D$  is strictly positive.

Define a map  $g : S^2 \rightarrow \mathbb{R}^2$  by setting

$$g(x) = \left( \frac{D_1(x)}{D(x)}, \frac{D_2(x)}{D(x)} \right).$$

Since the sets  $F_1, F_2, F_3$  cover  $S^2$ , for every point  $x$ , either  $D_1(x) + D_2(x) = D(x)$  or one of  $D_1(x)$  and  $D_2(x)$  is 0. Hence,  $g$  not only maps  $S^2$  into  $\mathbb{R}^2$ , but also into the boundary (perimeter) of the triangle with vertices  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$ . Let  $\varphi$  be a homeomorphism from this boundary to the circle  $S^1 \equiv \mathbb{R}/\mathbb{Z}$ , and let  $h = \varphi \circ g : S^2 \rightarrow \mathbb{R}, x \mapsto h(x) = \varphi(g(x))$ .

Let  $H : S^2 \rightarrow \mathbb{R}$  be such that  $h$  is the composition of  $H$  and the quotient map  $\mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \equiv S$ , and set  $K(x) = H(x) - H(f(x))$ . Note that if  $x_{\max}, y_{\min} \in S^2$  are such that  $H(x_{\max}) = \max H(x)$ , and  $H(y_{\min}) = \min H(x)$  then we have  $K(x_{\max}) \geq 0$  and  $K(y_{\min}) \leq 0$ . Since  $S^2$  is connected, this implies that  $K(x) = 0$  for some  $x$ , i.e.,  $H(x) = H(f(x))$ . But then  $g(x) = g(f(x))$ , which implies that one of the sets  $F_i$  contains both  $x$  and  $f(x)$ .  $\square$

**Notes.** This is a generalization of a theorem of Shkliarsky: it is due to Hadžiivanov and Nikiforov.

B. Bollobás, *Linear Analysis*, Cambridge University Press, (1990), xi + 240 pp.),

N. Hadžiivanov and V. Nikiforov, On the Borsuk–Ulam and Lyusternik–Shnirelman–Borsuk theorems (in Russian), *Annuaire Univ. Sofia Fac. Math. Inform.* **84** (1990), 83–89.

D. Shkliarsky, On subdivisions of the two-dimensional sphere (in Russian), *Mat. Sbornik*, **N. S.** **16** (1945), 125–128.

## 154. A Friendly Party: the Friendship Theorem of Erdős, Rényi and Sós

*Suppose that in a party of  $n \geq 3$  people, every two people have exactly one common friend. Then  $n$  is odd, one of the people is a friend of everybody, and everybody else has precisely two friends.*

*Proof.* Let  $G$  be the *friendship graph* in which the vertices are the people in the party and two acquaintances are joined by an edge. Note first that, as every edge is in a unique triangle, every degree is even, and the neighbourhood of a vertex of degree  $k$  contains precisely  $k/2$  edges (forming a complete matching on the neighbourhood). Furthermore, if  $G$  has a vertex of degree  $n - 1$ , then the entire graph  $G$  is just the union of  $(n - 1)/2$  edge disjoint triangles sharing a vertex, as claimed in the problem.

From now on, we assume that  $G$  has no vertex of degree  $n - 1$ , and our aim is to arrive at a contradiction.

First we show that any two non-adjacent vertices have the same degree. To see this, let  $a$  and  $b$  be non-adjacent vertices with a unique common neighbour  $c$ . Let  $A$  be the set of neighbours of  $a$  other than  $c$  and let  $B$  be the set of neighbours of  $b$  other than  $c$ . For every  $a' \in A$ , the vertices  $a'$  and  $b$  have a unique common neighbour, i.e.,  $a'$  has precisely one neighbour among the vertices in  $B$ . Similarly, every  $b' \in B$  has precisely one neighbour in the set  $A$ , since  $b'$  and  $a$  have a unique common neighbour. Hence the  $A - B$  edges of  $G$  give a one-to-one correspondence between  $A$  and  $B$ ; in particular  $a$  and  $b$  have the same degree. Consequently, vertices belonging to the same component of the complement  $\overline{G}$  of  $G$  have the same degree in  $G$ .

To complete the proof of regularity, let us show that  $\overline{G}$  is connected. Indeed, otherwise  $V(G) = U \cup W$  for some disjoint sets  $U$  and  $W$ ,  $|U| \leq |W|$ , such that every vertex in  $U$  is adjacent to every vertex in  $W$  in the graph  $G$ . Since  $G$  contains no quadrilateral, this implies that  $U$  consists of a single vertex and that vertex is adjacent to every other vertex, contrary to our assumption.

This proves that  $G$  is  $k$ -regular for some even  $k \geq 4$ , as the case  $k = 2$  is easily ruled out. Furthermore,  $n = k(k - 1) + 1$  since every vertex has  $k$  neighbours,  $k(k - 2)$  second-neighbours and no vertex at distance at least 3.

What we have done so far is pretty simple: we have just cleared the rubble and got to the real problem. So the solution starts just now: we have to show that for  $k \geq 3$  there is no  $k$ -regular friendship graph with  $n = k(k - 1) + 1$  vertices.

For a vertex  $x \in V(G)$ , let  $w_x(\ell)$  be the number of oriented closed walks of length  $\ell$  from  $x$  to  $x$ . Thus  $w_x(\ell)$  is the number of sequences of vertices  $x_0 x_1 \dots x_\ell$

such that  $x_0 = x = x_\ell$  and  $x_i x_{i+1}$  is an edge of  $G$  for every  $i$ ,  $0 \leq i < \ell$ . We claim that for  $\ell \geq 2$  we have

$$w_x(\ell) = k^{\ell-2} + (k-1)w_x(\ell-2). \quad (68)$$

Indeed, there are  $k^{\ell-2}$  walks of length  $\ell-2$  starting at  $x$ , with  $w_x(\ell-2)$  of these walks ending at  $x$ . Now, every such walk ending at  $x$  can be extended in  $k$  different ways to a walk of length  $\ell$  ending at  $x$ , and every other walk has precisely one such extension. This proves (68).

Since  $w_x(0) = 1$  and  $w_x(1) = 0$ , the recurrence (68) not only determines  $w_x(\ell)$  for every  $\ell$ , but also shows that it is independent of  $x$ :  $w_x(\ell) = w(\ell)$  for some function  $w$ . Clearly,  $w(0) = 1$ ,  $w(1) = 0$ ,  $w(2) = k$ ,  $w(3) = k$ , etc. Writing  $t(p)$  for the total number of closed walks of length  $\ell$  in  $G$ , we find that  $t(\ell) = nw(\ell)$ .

As  $k \geq 3$ , there is a prime  $p$  dividing  $k-1$ . For such a prime  $p$ , our recurrence (68) implies that

$$w(p) \equiv 1 \pmod{p}.$$

Since  $n = k(k-1) + 1 \equiv 1 \pmod{p}$ , we see that

$$t(p) = nw(p) \equiv w(p) \equiv 1 \pmod{p}. \quad (69)$$

To complete our solution, let us show that, contrary to (69), the total number  $t(p)$  of closed walks of length  $p$  is a multiple of  $p$ . Indeed, if  $W = x_0 x_1 \dots x_p$  is a closed walk (in particular,  $x_p = x_0$ ) then all  $p$  closed walks (with distinguished starting vertices)  $x_0 x_1 \dots x_p$ ,  $x_1 x_2 \dots x_p x_1$ ,  $x_2 x_3 \dots x_p x_1 x_2$ ,  $\dots$ ,  $x_{p-1} x_p x_1 \dots x_{p-2}$  obtained from  $W$  by starting it at various points are different closed walks. This contradiction completes our proof.  $\square$

**Notes.** The *Friendship Theorem* was published by Erdős, Rényi and Sós in 1966. Three years later Higman rediscovered it and reported his discovery in a lecture, which inspired Wilf to publish a proof.

As we have just seen, the reduction to the regular case is rather trivial, but eliminating the regular case is less so. The standard way of dealing with the regular case, as done by Wilf, goes via the Hoffman–Singleton theorem (see the original paper and also Bollobás). The beautiful proof above is due to Huneke, who apparently had found this elegant argument decades before he published it. On a slightly different note, Hammersley studied an infinite version of the problem.

B. Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics **184**, Springer-Verlag, (1998), xiv + 394 pp.

P. Erdős, A. Rényi and V.T. Sós, On a problem of graph theory, *Studia Sci. Math. Hungar.* **1** (1966), 215–235.

J.M. Hammersley, The friendship theorem and the love problem, in *Surveys in Combinatorics* (Southampton, 1983), London Math. Soc. Lecture Note Ser., **82**, Cambridge University Press, (1983), pp. 31–54.

G. Higman, Simple groups and combinatorial theory, lecture presented at the *Conference on Combinatorial Mathematics and Its Applications* in Oxford in 1969 (unpublished).

A.J. Hoffman and R.R. Singleton, On Moore graphs with diameters 2 and 3, *IBM J. Res. Develop.* **4** (1960), 497–504.

C. Huneke, The friendship theorem, *Amer. Math. Monthly* **109** (2002), 192–194.

H.S. Wilf, The friendship theorem, in *Combinatorial Mathematics and its Applications* (Proc. Conf., Oxford, 1969), Academic Press, (1971), pp. 307–309.

## 155. Polarities in Projective Planes

*Every polarity has a fixed point, i.e., a point which is mapped into a line through the point.*

*Proof.* Let  $\mathcal{P}$  be a projective plane with  $n^2 + n + 1$  points and  $n^2 + n + 1$  lines, with  $n + 1$  points on every line and  $n + 1$  lines through every point, where  $n \geq 2$ . Suppose that  $\phi : P \rightarrow L$  is a polarity that maps points into lines, and that no point is mapped into a line incident with the point. Define a graph  $G$  on  $P$  by joining a point  $v$  to every point on the line  $\phi(v)$ . Note that if  $v$  is joined to  $w$  then the point  $w$  is incident with the line  $\phi(v)$ , which implies that the line  $\phi(w)$  is also incident with the point  $\phi(\phi(v)) = v$ . Hence if  $v$  is joined to  $w$  then  $w$  is joined to  $v$ , and so  $G$  is indeed a graph (an unoriented graph!). Clearly,  $G$  is  $(n + 1)$ -regular, and any two of its vertices  $u$  and  $v$  have *precisely* one common neighbour, namely the point  $\phi^{-1}\ell$ , where  $\ell$  is the line through  $u$  and  $v$ . We are done by the Friendship Theorem in Problem 154: there is no such graph!  $\square$



## 156. Permutations of Vectors: Steinitz's Theorem

Let  $x_1, \dots, x_n$  be vectors of norm at most 1 in a  $d$ -dimensional normed space such that  $x_1 + \dots + x_n = 0$ . Then there is a permutation  $\pi \in S_n$  such that  $\|x_{\pi(1)} + \dots + x_{\pi(k)}\| \leq d$  for every  $k$ .

*Proof.* We shall prove a stronger assertion, since that suits induction better.

Let  $x_1, \dots, x_n$ ,  $n \geq d$ , be vectors of norm at most 1 in a  $d$ -dimensional normed space. Then there is a permutation  $\pi \in S_n$  such that

$$\left\| \sum_{i=1}^k x_{\pi(i)} - \frac{k-d}{n} x \right\| \leq d \quad (70)$$

for every  $k$ ,  $d \leq k \leq n$ , where  $x = \sum_{i=1}^n x_i$ .

Before proving this, we shall show the following Claim.

**Claim.** Let  $f_1, \dots, f_r, g_1, \dots, g_s : \mathbb{R}^k \rightarrow \mathbb{R}$  be linear functionals, and let

$$K = \{x \in \mathbb{R}^k : f_i(x) = a_i, i = 1, \dots, r; g_j(x) \leq b_j, j = 1, \dots, s\},$$

where  $a = (a_i) \in \mathbb{R}^r$  and  $b = (b_j) \in \mathbb{R}^s$ . Let  $x_0$  be a vertex (extreme point) of  $K$ , and set

$$J = \{j \in [s] : g_j(x_0) = b_j\}.$$

Then

$$|J| \geq k - r.$$

To check this Claim, note that if  $|J| + r < k$  then the system of equations

$$f_i(x) = 0, i = 1, \dots, r, g_j(x) = 0, j \in J$$

has a non-zero solution  $x_1 \in \mathbb{R}^k$ . But then, if  $\varepsilon > 0$  is sufficiently small, both of the vectors  $x_0 + \varepsilon x_1$  and  $x_0 - \varepsilon x_1$  belong to  $K$ , contradicting the extremality of  $x_0$ .

Armed with this claim, we turn to the proof of (70). By induction, we construct a chain of sets  $A_n = [n] \supset A_{n-1} \supset \dots \supset A_d$ , and numbers  $\lambda_k^i$ ,  $0 \leq \lambda_k^i \leq 1$  such that, for every  $k$ ,  $d \leq k \leq n$ , we have

$$|A_k| = k, \quad \sum_{i \in A_k} \lambda_k^i = k - d \quad \text{and} \quad \sum_{i \in A_k} \lambda_k^i x_i = \frac{k-d}{n} x.$$

To start the induction, note that for  $A_n = [n]$  we may take  $\lambda_n^i = \frac{n-d}{n}$  for every  $i$ .

Suppose we have found a set  $A_{k+1}$  and appropriate constants  $\lambda_{k+1}^i$ ,  $i \in A_{k+1}$ . Let  $K \subset \mathbb{R}^{k+1}$  be the set of points  $\mu = (\mu_i) \in \mathbb{R}^{k+1}$  such that

$$0 \leq \mu_i \leq 1, \quad \sum_{i \in A_k} \mu_i = k - d, \quad \text{and} \quad \sum_{i \in A_k} \mu_i x_i = \frac{k-d}{n} x.$$

Note that  $K \subset \mathbb{R}^{k+1}$  is a compact and convex polyhedron given by  $d+1$  equations and  $2(k+1)$  inequalities.

The set  $K$  is not empty, since it contains the point  $\mu = (\mu_i)$  with  $\mu_i = \frac{k-d}{k+1-d} \lambda_{k+1}^i$ ,  $i \in A_{k+1}$ . Let  $v = (v_i)$  be an extremal point of  $K \subset \mathbb{R}^{k+1}$ . By our Claim,

$$|\{i : v_i \text{ is 0 or 1}\}| \geq k+1 - (d+1) = k-d.$$

If no  $v_i$  were 0, this would imply the contradiction  $\sum_{i \in A_k} v_i > k-d$ ; hence,  $v_h = 0$  for some  $h \in A_{k+1}$ . Consequently,  $A_k = A_{k+1} \setminus \{h\}$  will do, completing the proof of the existence of the sets  $A_n, A_{n-1}, \dots, A_d$ .

Finally, define  $\pi(k) = A_k \setminus A_{k-1}$  for  $k = d+1, \dots, n$ , and  $A_d = \{\pi(1), \dots, \pi(d)\}$ . Then for  $d \leq k \leq n$  we have  $\sum_{i=1}^k x_{\pi(i)} = \sum_{i \in A_k} x_i$ , and so

$$\left\| \sum_{i=1}^k x_{\pi(i)} - \frac{k-d}{n} x \right\| = \left\| \sum_{i \in A_k} (1 - \lambda_k^i) x_i \right\| \leq \sum_{i \in A_k} (1 - \lambda_k^i) = k - (k-d) = d.$$

□

**Notes.** For a finite-dimensional normed space  $E$ , write  $S = S(E)$  for the minimal real number such that if  $x_1, \dots, x_n \in E$  have norm at most 1 then for some permutation  $\pi \in S_n$  we have  $\|\sum_{i=1}^k x_{\pi(i)}\| \leq S$ . Needless to say, *a priori* it is not clear that there is such a (finite) constant: this was proved by Steinitz in 1913 (see p. 171), and in his honour  $S(E)$  is called the *Steinitz constant* of the space  $E$ . In fact, Steinitz showed that if  $E$  is a  $d$ -dimensional Euclidean space then  $S(E) \leq 2d$ .

The greatly improved estimate in the proof above, namely that  $S(E) \leq d$ , was found by Grinberg and Sevastyanov in 1980: in the proof above we closely followed their proof. In fact, between these two results numerous papers were published giving bounds on  $S(E)$  that are worse than Steinitz's original bound.

The proof of Grinberg and Sevastyanov shows more than the inequality  $S(E) \leq \dim(E)$  we claimed above, since it does not make use of the symmetry of the norm (that  $\| -x \| = \|x\|$  for every  $x$ ), so it applies to every positive subadditive functional. (Recall that  $p : \mathbb{R}^d \rightarrow \mathbb{R}_+$  is a positive subadditive functional if  $p(tx) = tp(x)$  whenever  $x \in \mathbb{R}^d$  and  $t \geq 0$ , and  $p(x+y) \leq p(x) + p(y)$  for all  $x, y \in \mathbb{R}^d$ ; see Bollobás, p. 48.) This proof of Grinberg and Sevastyanov that we have just presented shows that, with the obvious definition of  $S(p)$ , if  $p$  is a

positive subadditive functional on a  $d$ -dimensional space then  $S(p) \leq d$ . Grinberg and Sevastyanov pointed out that this bound is best possible: if the ‘unit ball’  $\{x : p(x) \leq 1\}$  of  $p$  is a  $d$ -simplex whose vertex vectors sum to 0, then  $S(p) = d$ , since  $S(p) \geq d$  is implied by the sequence of  $d + 1$  vertices.

Concerning Euclidean spaces, Grinberg and Sevastyanov pointed out that  $S(E_d) \geq (d + 3)^{1/2}/2$  for dimension  $d$ , and Banaszczyk proved that for  $d = 2$ , the plane, we have equality here.

For numerous further results in the area, including extensions to infinite-dimensional spaces, see the papers listed below.

W. Banaszczyk, The Steinitz constant of the plane, *J. Reine Angew. Math.* **373** (1987), 218–220.

W. Banaszczyk, The Steinitz theorem on rearrangement of series for nuclear spaces, *J. Reine Angew. Math.* **403** (1990), 187–200.

W. Banaszczyk, A note on the Steinitz constant of the Euclidean plane, *C. R. Math. Rep. Acad. Sci. Canada* **12** (1990), 97–102.

W. Banaszczyk, Rearrangement of series in nonnuclear spaces, *Studia Math.* **107** (1993), 213–222.

I. Bárány and V. Grinberg, On some combinatorial questions in finite-dimensional spaces. *Linear Algebra Appl.* **41** (1981), 1–9.

I. Bárány and V. Grinberg, A vector-sum theorem in two-dimensional space, *Period. Math. Hungar.* **16** (1985), 135–138.

V. Bergström, Zwei Sätze über Vektorpolygone, *Abh. Math. Se. Hamb. Univ.* **8** (1931), 206–214.

B. Bollobás, *Linear Analysis*, Cambridge University Press, (1990), xi + 240 pp.

J. Bonet and A. Defant, The Lévy-Steinitz rearrangement theorem for duals of metrizable spaces, *Israel J. Math.* **117** (2000), 131–156.

V.S. Grinberg and S.V. Sevastyanov, The value of the Steinitz constant, (in Russian), *Funktsional. Anal. i Prilozhen.* **14** (1980), 56–57; English translation: *Functional Anal. Appl.* **14** (1980), 125–126.

H. Hadwiger, Ein Satz über geschlossene Vektorpolygone des Hilbertschen Raumes, *Math. Z.* **41** (1936), 732–738.

H. Hadwiger, Über das Umordnungsproblem im Hilbertschen Raum, *Math. Z.* **46** (1940), 70–79.

H. Hadwiger, Ein Satz über bedingt konvergente Vektorreihen, *Math. Z.* **47** (1942), 663–668.

I. Halperin, Sums of a series, permitting rearrangements, *C. R. Math. Rep. Acad. Sci. Canada* **8** (1986), 87–102.

V.M. Kadets, On a problem of the existence of convergent rearrangement, (in Russian), *Izv. Vyssh. Uchebn. Zaved. Mat.* **3** (1992), 7–9.

V. Klee, Rearrangements of series of vectors, *Math. Z.* **81** (1963), 46–51.

E. Steinitz, Bedingt konvergente Reihen und Konvexe Systeme, *J. Reine Angew. Math.* **143** (1913), 128–175; **144** (1914), 1–40; **146** (1916), 1–52.

## 157. An American Story

*The twenty Death Row prisoners have a strategy which gives them a better than even chance of escaping execution.*

*Proof.* Here is a suitable strategy for the prisoners. They assign the boxes to themselves at random, so that every one of them has his ‘own’ box. Needless to say, this 1–1 assignment has nothing to do with the names in the boxes. Every prisoner first opens his ‘own’ box. If he finds his name, he may as well stop. Otherwise, next he opens the box assigned to the prisoner whose name he finds there, and so on. To spell this out, writing  $P_1, \dots, P_{20}$  for the names of the prisoners, and  $B_1, \dots, B_{20}$  for the boxes, if  $B_5$  contains  $P_8$ ,  $B_8$  contains  $P_3$ ,  $B_3$  contains  $P_{19}$ , and  $B_{19}$  contains  $P_5$ , then prisoner  $P_5$  first opens  $B_5$ , then  $B_8$ ,  $B_3$  and  $B_{19}$ , upon which he gives a sigh of relief, as he has just found his own name. With this,  $P_5$  may as well stop, or else open another eight boxes out of curiosity.

All that remains is to check that this strategy succeeds with probability at least  $1/2$ . To this end, note that the random assignment of boxes to prisoners is equivalent to a permutation of  $B_1, \dots, B_{20}$ : to obtain this permutation, map  $B_i$  into  $B_j$  if  $B_i$  contains  $P_j$ . Then the strategy of the prisoners is to open boxes along cycles: prisoner  $P_i$  finds his name if the cycle containing  $B_i$  has length at most 12. Hence, the prisoners gain their reprieve if the permutation they take at random contains no long cycle, i.e., no cycle of length at least 13.

It is easy to calculate the probability  $p_L$  that our random permutation of length 20 contains a long cycle. Indeed, as a permutation contains at most one long cycle,  $p_L$  is just the expected number of long cycles. Now, the expected number of cycles of length  $\ell \geq 2$  in a random permutation of  $[n] = \{1, \dots, n\}$  is

$$\binom{n}{\ell} \frac{\ell!}{\ell} (n - \ell)! / n! = \frac{1}{\ell},$$

since there are  $\ell!/\ell$  oriented cycles on  $\ell$  given vertices. Consequently,

$$p_L = \frac{1}{13} + \frac{1}{14} + \dots + \frac{1}{20} < 0.5,$$

as claimed. □

**Notes.** This problem was posed by Peter Bro Miltersen in a paper with Anna Gál; the unexpected and beautiful solution above was given by Sven Skyum. The problem reached me during a meeting in Oberwolfach attended by both Gál and Miltersen.

In a lovely article about the problem, Curtin and Warshauer pointed out that the strategy above is best possible. Indeed, let us change our game giving the prisoners more chance. Let the first prisoner open boxes in some order, stopping when he finds his own name. The next man to go into the room is the prisoner with the *minimal index* among those whose names have not been found by the first prisoner. He is told *not to look* among the names found by the first man, and to stop his search when he finds his name and not before. The third man to go into the room is the one whose index is minimal among those whose names have not been found by the previous prisoners; he is told where not to look, and so on. The prisoners win (are reprieved) if none of them opens more than 12 boxes.

Clearly, in this altered game the prisoners have at least as much chance of winning as in the original game. Also, as the order of the boxes is random, no matter what strategy the prisoners use in the second game, they win if and only if the permutation they construct has no cycle of length at least thirteen. Hence Skyum's strategy is indeed optimal.

E. Curtin and M. Warshauer, The locker puzzle, *Mathematical Intelligencer* **28** (2006), 28–31.

A. Gál and P.B. Miltersen, The cell probe complexity of succinct data structures. In *Automata, Languages and Programming*, 332–344, Lecture Notes in Comput. Sci. **2719**, Springer, Berlin, 2003.

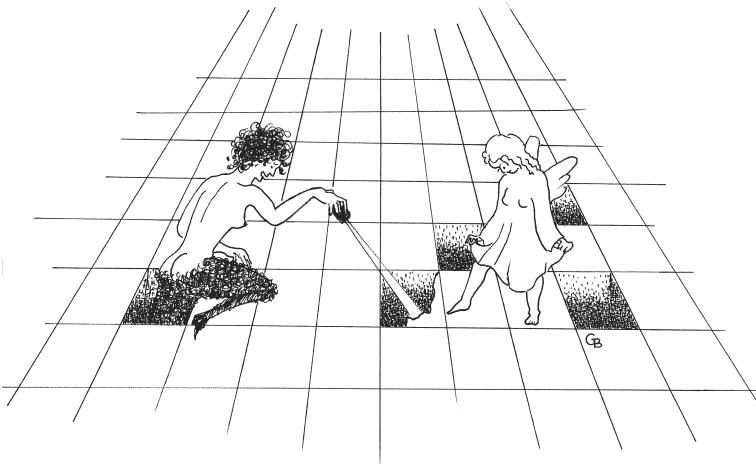
## 158. Conway's Angel and Devil Game

*The Angel of speed 5 defeats the Devil.*

*Proof.* First, let us clarify how the game is played. Both A, the Angel, and D, the Devil, play according to strategies which, given the development of the game so far, tells them what to do next. If after  $t - 1$  steps the Angel has travelled along  $a_0 = 0, a_1, \dots, a_{t-1}$ , and the Devil has devoured the squares  $d_1, \dots, d_{t-1}$ , then first the Devil eats a square  $d_t = \delta(a_1, \dots, a_{t-1}; d_1, \dots, d_{t-1})$ , and then the Angel jumps to a square  $a_t = \alpha(a_1, \dots, a_{t-1}; d_1, \dots, d_t)$ . The function  $\alpha$  is the *strategy* of A and  $\delta$  is the *strategy* of D. The restriction on  $\delta$  is that  $d_t \neq a_{t-1}$ , and the restrictions on  $\alpha$  are that  $a_t \notin \{d_1, \dots, d_t\}$  and  $d(a_t, a_{t-1}) \leq c$ .

We *could* assume that not only do we have  $d_t \neq a_{t-1}$  but also  $d_t \neq d_i$  for  $i < t$ : after all, the Devil has to be rather stupid to eat a square she has already eaten. However, there is no need to make this assumption. In fact, it is convenient to give the Devil more (useless) freedom by allowing him to eat *nothing* if he so wishes, i.e., to *pass* on his move.

Needless to say, the functions  $\alpha$  and  $\delta$  determine the flow of the entire game, i.e., the infinite sequences  $a_0 = 0, a_1, \dots$  and  $d_1, d_2, \dots$ .



The Devil and the Angel

The Angel wins if she has a strategy  $\alpha$  such that no matter what strategy  $\delta$  the Devil plays (what moves she plays), the sequence  $a_0, a_1, \dots$  is unbounded. The Devil wins if she has a strategy  $\delta$  such that no matter what strategy the Angel

chooses (what moves she plays), her route is bounded. It is easy to see that precisely one of the players has a winning strategy.

We write  $A \rightarrow D$  for the statement that A beats D; also,  $A(\alpha) \rightarrow D$  means that  $\alpha$  is a winning strategy for A. We shall freely use quantitative self-explanatory variants of these statements. For example,  $A \xrightarrow{n} D$  means that A can get out of the box  $B_n = \{(x_1, x_2) \in \mathbb{Z}^2 : \max |x_i| \leq n\}$ , and  $A_c(\alpha) \xrightarrow{n,T} D$  means that the Angel of speed  $c$  can get out of the box  $B_n$  in time  $T$  if she plays strategy  $\alpha$ . Note that if  $A \xrightarrow{n} D$  then  $A \xrightarrow{n,T_n} D$  for  $T_n = (2n + 1 + 2c)^2 = |B_{n+c}|$ , since if the only aim of D is to prevent A from getting out of  $B_n$ , then she will eat only squares in  $B_{n+c}$ . Therefore, we take  $A \xrightarrow{n} D$  to mean that for some strategy  $\alpha_n$ , A gets out of  $B_n$  within time  $T_n$ . Clearly, for that one needs only a strategy of ‘length’  $T_n$ .

Trivially, if  $A \rightarrow D$  then  $A \xrightarrow{n} D$  for every  $n$ . The converse is only a little less obvious: indeed, it follows from a standard compactness argument.

**Claim 1.** *If  $A \xrightarrow{n} D$  for every  $n$  then  $A \rightarrow D$ .*

*Proof.* To see this, we build a strategy  $\alpha^*$  from initial segments of winning strategies for the Angel for getting out of infinitely many of the boxes  $B_n$ .

For each  $n$ , let  $\alpha_n$  be such that  $A(\alpha_n) \xrightarrow{n} D$ . From the initial position, infinitely many of these strategies  $\alpha_n$  agree on the Angel’s first move: say, for some infinite set  $I_1$ , the strategies  $\alpha_i$ ,  $i \in I_1$ , all require the Angel to move to a position  $a_1^*$ . So in the strategy  $\alpha^*$  the Angel moves to  $a_1^*$ . Given the Devil’s reply to this move, infinitely many of the  $\alpha_i$ ,  $i \in I_1$ , agree on the Angel’s reply: say, for some infinite subset  $I_2$  of  $I_1$ , the strategies  $\alpha_i$ ,  $i \in I_2$ , all require the Angel to move to  $a_2^*$ . So in the strategy  $\alpha^*$  the Angel moves to  $a_2^*$ . Continuing in this way, the Angel gets arbitrarily far from the origin.  $\square$

Henceforth, our aim is to prove that  $A \xrightarrow{n} D$  for every  $n$ .

Let us call a strategy  $\alpha$  of A *simple* if it never makes A jump to a square she could have jumped to earlier, i.e.,  $d(a_{t+1}, a_i) > c$  for all  $i < t$ . Also, a strategy  $\delta$  of D is *simple* if at his  $(t+1)$ st step, when A has constructed a sequence  $a_0, a_1, \dots, a_t$ , the Devil does not eat a square within distance  $c$  of any of the squares  $a_0, a_1, \dots, a_{t-1}$ , i.e.,  $d(d_{t+1}, a_i) > c$  for every  $i < t$ . (Note that  $d(d_{t+1}, a_t) \leq c$  is allowed.) The *Simple Angel*, SA, is an Angel (of speed  $c$ ) bent on using a simple strategy, and the *Simple Devil*, SD, is defined analogously.

**Claim 2.** *If  $A \xrightarrow{n} SD$  then  $SA \xrightarrow{n} SD$ .*

*Proof.* Let  $A(\alpha) \xrightarrow{n} SD$ . We construct a winning simple strategy  $\alpha_s$  from  $\alpha$  by altering the sequence  $a_0, a_1, \dots$  step by step, from the beginning. Suppose that we

have defined the first  $t$  moves of the Angel:  $a_0, a_1, \dots, a_t$ , and D has devoured his squares up to  $d_{t+1}$ . To define the Angel's next move in the simple strategy  $\alpha_s$ , take a continuation  $a_{t+1}, a_{t+2}, \dots, a_u$  that brings A back to within  $c$  of  $a_t$  as late as possible; this maximum exists since the entire sequence  $a_1, a_2, \dots$  has at most  $T_n$  terms. Ignoring all intermediate moves,  $\alpha_s$  makes the Angel jump from  $a_t$  to  $a_u$ , making it the new  $a_{t+1}$ . Note that this  $a_{t+1}$  (the old  $a_u$ ) is at distance greater than  $c$  from each  $a_i$ ,  $0 \leq i \leq t-1$ , since otherwise this late step  $a_u$  would have been used as  $a_{i+1}$  to continue the sequence  $a_0, \dots, a_i$ . Continuing in this way, we obtain a new winning strategy,  $\alpha_s$ , which is simple:  $A(\alpha_s) \xrightarrow{n} SD$ .  $\square$

The alert reader is bound to wonder why we had  $SD$  rather than  $D$  in the Claim above; the answer is that we shall need the assertion for  $SD$ , rather than  $D$ : the argument above proves also that  $A \xrightarrow{n} D$  implies  $SA \xrightarrow{n} D$ .

**Claim 3.** If  $SA \xrightarrow{n} SD$  then  $SA \xrightarrow{n} D$ .

*Proof.* This is even more obvious than the previous Claims. Indeed, let  $\alpha_s$  be a simple strategy such that  $A(\alpha_s) \xrightarrow{n} SD$ . Let A employ the same simple strategy against D. In this case, every move of D that does not satisfy the condition for SD is simply *wasted*: after time  $t$ , whether or not D has eaten any of the squares within  $c$  of  $a_0, a_1, \dots, a_{t-1}$ , the Angel has no intention of using them for her future moves. Hence  $A(\alpha_s) \xrightarrow{n} SD \Rightarrow A(\alpha_s) \xrightarrow{n} D$ , as required.  $\square$

**Claim 4.** If  $A \rightarrow SD$  then  $A \rightarrow D$ .

*Proof.* This is immediate from Claims 1–3.  $\square$

By this Claim, it suffices to show that  $A \rightarrow SD$ . Although so far we have done very little, this reduction of the original problem  $A \rightarrow D$  to  $A \rightarrow SD$  does make our task very easy.

In fact, we shall prove more: even if the Angel ‘runs’ in a rather restricted way, she still defeats the Simple Devil. To set the scene, we change our description of the game. When the Devil (in our case, this will be the Simple Devil) blows up a square, then she also builds a high and impenetrable wall of negligible width around the hole she has created. (Thus a square the Simple Devil has blown up is a  $1 \times 1$  hole surrounded by a wall, as in Fig. 90.)

The *Running Angel*, RA, is then defined as follows.

1. She is rather generous: she gives the squares in the negative half-plane to the Devil, i.e., deems them to have been devoured before the game began.
2. Starting from the square  $0 = (0, 0)$ , as before, so that on her left there is an infinite wall, RA stretches out her left hand to touch the wall, and keeping



her left hand on the wall, she runs as far as possible, i.e., until her next step (to a square in which she touches the wall on her left) would take her more than distance  $c$  away from her starting position. She repeats this very simple strategy at every time step, continuing in the direction of her last step. If she cannot get  $c + 1$  away from her starting position, i.e., from  $a_{t-1}$  at time  $t$ , so that she would not know when to stop, then she forfeits the game. However, as we shall see shortly, this never happens.

Figure 89 shows a route of RA from  $a_{t-1}$  to  $a_t$ .

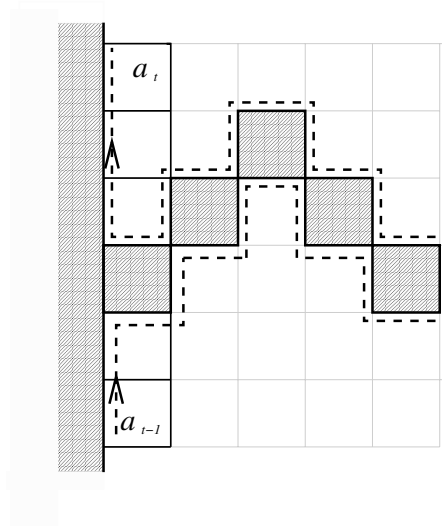


Fig. 89. In step  $t$ , from  $a_{t-1} = (0, 3)$ , with the squares  $(0, 5)$ ,  $(1, 6)$ ,  $(2, 7)$ ,  $(3, 6)$  and  $(4, 5)$  blown up by the Devil, the speed 5 Running Angel goes to  $a_t = (0, 8)$ .

To show that  $RA \rightarrow SD$ , we add a final colour to our description to the flow of the game. As RA runs along the wall, rather than touching the wall with her left hand, she holds a brush in her left hand and paints a *green stripe* on the wall. We need the following Claim (three simple assertions) about the Running Angel.

**Claim 5.** Suppose that the entire run  $a_0 \dots a_1 \dots a_2 \dots \dots a_t$  of RA up to time  $t$  is strictly in the upper half plane, with the exception of  $a_0 = 0$ . Then:

- (i) no panel is painted twice;
- (ii) in every step, at least  $c$  panels get painted;
- (iii) the height of  $a_t$  is at least  $(c - 4)t$ .

*Proof.* To see (i), simply check that the first stretch of stripe to be painted twice cannot arise, as there must be *two* squares preceding it with walls on their left, and

SD is not allowed to blow up a square within distance  $c \geq 2$  of an earlier position of the Angel. Figure 90 shows the three possibilities that can arise if  $xy$  is the first edge to be painted twice.

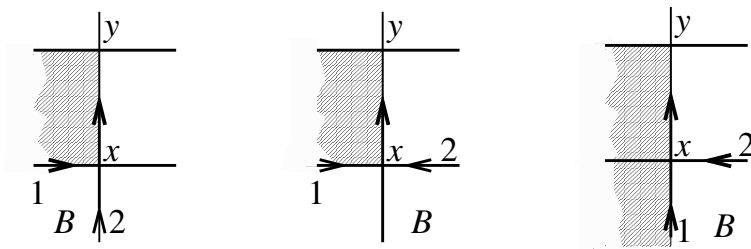


Fig. 90. If the edge  $xy$  was first painted after edge 1, and then again after edge 2, three possibilities arise. When edge 2 was painted, the square marked  $B$  had to be blown up, but as  $B$  is within distance 2 of the square on the left of the edge marked 1, this is a contradiction.

Part (ii) is immediate from (i).

Finally, part (iii) follows by simple accounting. As RA runs from  $a_0$  to  $a_t$ , she paints at least  $ct$  panels. Much of the time she paints the original straight-line infinite wall  $W$ , the  $y$  axis, *upwards*, increasing her height. This should take her height to  $ct$ .

When RA does not paint the original wall  $W$ , she paints the walls of the squares that have been blown up up to then. What happens to the height during those moves? Whenever a horizontal side of a blown-up square is painted, the height of the Angel does not change: at most  $2t$  steps are wasted.

Also, painting the right side of a blown-up square increases the height by one, so is just as good as painting  $W$ .

Finally, painting the left side of the wall of a square not only wastes a step (decreases the number of panels of  $W$  that will be painted by RA while she paints  $ct$  panels), but also decreases the height by one.

Putting it all together, we find that

$$y(a_t) \geq ct - 2t - t - t = (c - 4)t,$$

proving (iii). □

Armed with assertions (ii) and (iii), the solution of the problem is easily completed. Note that during her first  $c - 1$  moves, the rise of the Running Angel cannot be slowed down by the Simple Devil (or the Devil himself): with every move, RA will increase her height by the maximum,  $c$ . Hence,  $y(a_{c-1}) = c(c - 1)$ . Now, let us take  $c$  to be 5. Then  $y(a_4) = 20$  so, *a fortiori*,  $y(a_{10}) \geq 10$ , since the 10 blown-up squares cannot lower the height by more than 10 during the run from  $a_4$

to  $a_{10}$ . This implies that during her journey from  $a_0$  to  $a_{11}$ , RA never sinks below height  $10 - 5 = 5$ ; consequently, by (ii), up to time 11, RA does not repaint a wall and so, by (iii),  $y(a_{11}) \geq (5 - 4)11 = 11$ . This pattern is repeated for every journey from time  $t - 1 \geq 11$  to time  $t$ : running from  $a_0$  to  $a_t$ , RA stays in the upper half plane, and so  $y(a_t) \geq t$ . In particular, the height of the Running Angel tends to  $+\infty$ , showing that she beats the Simple Devil, and completing the proof.  $\square$

**Notes.** Conway's Angel-and-Devil problem was solved independently by Bowditch, Máthé, Gács, and Kloster in 2006, after it had been open for about thirty years. The solution above is essentially that of Máthé.

The precursor of Conway's game is the game *Quadruphage* invented in the 1960s and popularized by Epstein and Silverman in their books on games. Quadruphage is played on a finite (say,  $(2k + 1) \times (2k + 1)$ ) squared board by the *King* and the *Quadruphage* (square eater). The King is just the Angel of speed 1, and the Quadruphage is the Devil; the King starts from the centre, and the players alternate, as above. The King wins if she can reach the boundary and so escape from the Quadruphage. It is easily seen that the King wins on a small board, and the Quadruphage on a large one.

It was this game that led Conway to ask in the 1970s whether the Angel of sufficient speed can escape from the Devil on the infinite board  $\mathbb{Z}^2$ . (As the Quadruphage wins on a large board, the Devil beats the Angel of speed 1.) In 1982, Berlekamp, Conway and Guy were the first to give an account of this game in print. They also stated that T.W. Körner had shown that in the game played on  $\mathbb{Z}^d$  instead of  $\mathbb{Z}^2$ , if the dimension  $d$  and the speed  $c$  are sufficiently large then the Angel of speed  $c$  can escape. However, there seems to be no trace of the proof of this result.

About thirty years after the invention of this problem, Kutz, and Bollobás and Leader gave winning strategies for an Angel in dimension  $d = 3$ : the former for an Angel of speed 13, and the latter for an Angel of speed 50. The proofs are not too dissimilar: they are based on keeping track of the densities of eaten cubes at larger and larger scales (the 'danger' they represent to an Angel trying to cross them) and planning the escape route on infinitely many levels.

Bollobás and Leader also considered the game of 'Angel and Infinitely Many Devils'. In this variant of the game we are given as parameters a speed  $c$  and some distances  $n_0, n_1, \dots$ , with  $n_0 = 0$ . On the Angel's turn, she as usual jumps to any square at distance at most  $c$  from his current position. On the Devils' turn, the Devils eat a family of squares: one square at distance at least  $n_0$  from the Angel, another square at distance at least  $n_1$ , and so on. As before, the Angel wins if she can get arbitrarily far from the origin. The question is: does there exist a speed  $c$  and a sequence  $n_0, n_1, \dots$  so that the Angel wins? In this version, it is easy to see

that the Devils win in dimension 2, and Bollobás and Leader showed that the Angel wins in dimensions at least 3.

Turning to Conway's original problem – whether in the game played on  $\mathbb{Z}^2$  an Angel of sufficient speed can defeat the Devil – around 1990 Conway made the observation that, in the notation used in the solution above, if  $A \rightarrow D$  then  $SA \rightarrow D$ . In other words, the Angel's strategy may as well be simple. Unfortunately, this observation did not seem to bring a solution any nearer.

Amazingly, if we turn around this observation and prove that if  $A \rightarrow SD$  then  $A \rightarrow D$ , as in Claim 4, then a solution stares us in the face. This is exactly what Bowditch and Máthé have done to come up with their solutions, and this is at the heart of the solution we gave above. In fact, our presentation is very close to that of Máthé, including the use of a Running Angel and the green stripe painted by her. The proofs of Gács and Kloster proceed along very different lines. Although we shall not go into the details, we remark that Kloster's solution is totally audacious: he takes a route that clearly cannot lead anywhere – but it does.

In the solution above, speed 5 is not best possible; in fact, with the tiniest amount of extra care one can show that the Angel of speed 4 also beats the Devil. Mathé and Kloster proved that the Angel of speed 2 wins. (As we noted earlier, this is best possible: the Angel of speed 1 loses.) In fact, this needs only a slight strengthening of the Máthé–Bowditch solution above: we just have to be slightly less cavalier in our instructions to the Running Angel. Indeed, if we get the Devil to build walls around the largest regular *octagon* inscribed in each blown-up square (rather than around the square itself), so that RA can squeeze through the gap between two blown-up squares sharing only a vertex, then, as shown by Mathé, it is not too difficult to prove that RA of speed 2 (hugging the walls of the octagons) beats SD.

E.R. Berlekamp, J.H. Conway and R.K. Guy, *Winning Ways for your Mathematical Plays*, Volume 2, Academic Press, 1982, xxxi+850pp.; see also Second Edition, Volume 3, A.K. Peters, 2003, p. 643.

B. Bollobás and I. Leader, The Angel and the Devil in three dimensions, *Journal of Combinatorial Theory, Series A* **113** (2006), 176–184.

B.H. Bowditch (2006) The angel game in the plane, *Combinatorics, Probability and Computing* **16** (2007)

J.H. Conway, The Angel problem. In *Games of No Chance* (Nowakowski, R.J., ed.), Cambridge University Press, 1996, pp. 3–12.

R.A. Epstein, *Theory of Gambling and Statistical Logic*, Academic Press, 1967; Revised Edition, 1995, xv + 450 pp.

P. Gács, The angel wins, *Manuscript*, 2006.

O. Kloster, A solution to the angel problem, *Manuscript*, 2006.

M. Kutz, Conway's Angel in three dimensions, *Theoret. Comput. Sci.* **349** (2005), 443–451.

M. Kutz and A. Pór, Angel, Devil, and King. In *Computing and Combinatorics*, Lecture Notes in Comput. Sci. **3595** (2005), Springer, Berlin, pp. 925–934.

A. Máthé, The angel of power 2 wins, *Combinatorics, Probability and Computing* **16** (2007)

D.L. Silverman, *Your Move – Logic, Math and Word Problems for Enthusiasts*, McGraw Hill, 1971; Dover Publications, 1991, 221 pp.

## 159. The Point–Line Game

*Line has a winning strategy for the Point–Line Game: playing in a suitable way, it can force Point to construct a convergent sequence.*

*Proof.* The winning strategy for Line is based on the existence of a family of subsets of the unit disc  $D$ .

**Claim.** *There is a nested family  $\{K_\gamma : \gamma \in \Gamma\}$  of compact convex subsets of  $D$  with index set  $\Gamma \subset [0, \pi]$  such that*

- (i)  $\Gamma$  is dense in  $[0, \pi]$  and  $K_\gamma$  has area  $\gamma$  for each  $\gamma \in \Gamma$ ,
- (ii) if  $0 \leq \gamma_1 < \gamma_2 \leq \pi$  then  $K_{\gamma_1} \subset K_{\gamma_2}$ ,
- (iii) for any  $0 < c \leq \pi$  and  $d > 0$  there exist  $\gamma, \delta$  with  $\gamma < c \leq \delta$  such that  $K_\delta \setminus K_\gamma$  has diameter at most  $d$ .
- (iv) for any  $c \in \Gamma$  we have  $\bigcap_{\gamma > c} K_\gamma = K_c$ .
- (v)  $K_0 = \{0\}$ .

Let us see first how to complete the proof from this claim. If Point always plays the point 0 then the sequence is trivially convergent. Thus, we may assume that the first point picked by Point is not 0.

We shall write  $(x_n)$  for the sequence of points picked by Point, and  $(\ell_n)$  for the sequence of lines constructed by Line. Thus, Line constructs a line  $\ell_n$  through  $x_n$ , and Point chooses a point  $x_{n+1}$  on  $\ell_n$ .

As our task is to show that Line wins, we shall play its role. We, as Line, choose two sequences of indices form  $\Gamma$ ,  $\gamma_0 = 0 \leq \gamma_1 \leq \dots \leq \pi$  and  $0 \leq \delta_1, \delta_2, \dots \leq \pi$  such that, for every  $n \geq 1$ ,

- (a)  $0 \leq \delta_n - \gamma_n \leq 1/n$ ;
- (b) the set  $D_n = K_{\delta_n} \setminus K_{\gamma_n}$  has diameter at most  $1/n$ ;
- (c)  $x_n \in D_n$ .

To prove that we *can* choose such sequences of indices, suppose that we have found appropriate indices  $\gamma_0 \leq \dots \leq \gamma_{n-1}$  and  $\delta_1, \dots, \delta_{n-1}$  for the points  $x_1, \dots, x_{n-1}$ , and  $x_n \notin K_{\gamma_{n-1}}$ . Set

$$c_n = \inf\{\gamma : K_\gamma \text{ contains } x_n\}.$$

By condition (iv), if  $\gamma \in \Gamma$  is at least  $c_n$  then  $x_n \in K_\gamma$ . Using (iii), Line chooses  $\gamma_n$  and  $\delta_n$  such that with  $D_n = K_{\delta_n} \setminus K_{\gamma_n}$  we have  $x_n \in D_n$  and  $\delta_n - \gamma_n < 1/n$ , and  $D_n$  has diameter less than  $1/n$ . Moreover, since  $x_n \notin K_{\gamma_{n-1}}$ , we may demand that  $\gamma_n \geq \gamma_{n-1}$ . Now, for the line  $\ell_n$  we choose a line through  $x_n$  that is disjoint from the closed convex set  $K_{\gamma_n}$ . For example, writing  $y_n$  for the point of  $K_{\gamma_n}$  nearest to  $x_n$ , for  $\ell_n$  we may take the line through  $x_n$  that is perpendicular to  $x_n y_n$ . (To sound

more highbrow, we can appeal to the Hahn-Banach Theorem to find  $\ell_n$ .) Wherever Point chooses  $x_{n+1}$ , it will not be in  $K_{\gamma_n}$  and so the strategy continues.

We claim that with this strategy, Line forces the sequence  $(x_n)$  to be convergent, and so wins the Point–Line Game. To prove this, it suffices to show that for every positive integer  $M$  there is a set of diameter at most  $1/M$  that contains  $x_n$  for all sufficiently large  $n$ . Indeed, then  $(x_n)$  is a Cauchy sequence, and so convergent.

Set  $c = \sup \gamma_n$ . Suppose that there is an  $N \geq M$  with  $\delta_N > c$ . In this case, since  $\delta_n \rightarrow c < \delta_N$ , if  $n \geq N$  is sufficiently large then  $\delta_n < \delta_N$ , so  $x_n$  is in the set  $K_{\delta_n} \setminus K_{\gamma_n} \subset K_{\delta_N} \setminus K_{\gamma_N} = D_N$ , which is a set of diameter less than  $1/N \leq 1/M$ .

Consequently, we may assume that  $\delta_n \leq c$  for every  $n \geq M$ . By (iii), there are  $\gamma < c \leq \delta$  such that  $K_\delta \setminus K_\gamma$  has diameter at most  $1/M$ . If  $n$  is sufficiently large then  $\gamma < \gamma_n \leq c$  and so  $x_n \in K_{\delta_n} \setminus K_{\gamma_n} \subset K_\delta \setminus K_\gamma$ . Consequently,  $(x_n)$  is indeed a Cauchy sequence, so Line does win.

It remains to prove the Claim. This is, in fact, very easy, although the assertion is somewhat surprising. Starting with  $K_\pi = D$ , we shall obtain the sets  $K_\gamma$  in countably many rounds. Note that, having chosen a set, we do not have to bother with defining its suffix  $\gamma$ , since that is just the area of the set.

First, let us construct a countable system of sets  $K_\gamma$  by cutting off from  $D$  the half-moon denoted by 1 in Figure 91; then the half-moons 1 and 2; then 1, 2 and 3; then the three half-moons 1, 2, 3, and the triangle 4; and so on. So far, the diameter of a difference  $K_\gamma \setminus K_{\gamma'}$  is at most  $\sqrt{3}$ . Now, how can we cut up a half-moon or a triangle into pieces, so that each cut has small diameter?

Suppose first that we are about to cut off a triangle, i.e., a compact convex set  $K_\gamma$  we have already defined contains a triangle  $T = ABC$  ‘on its boundary’;

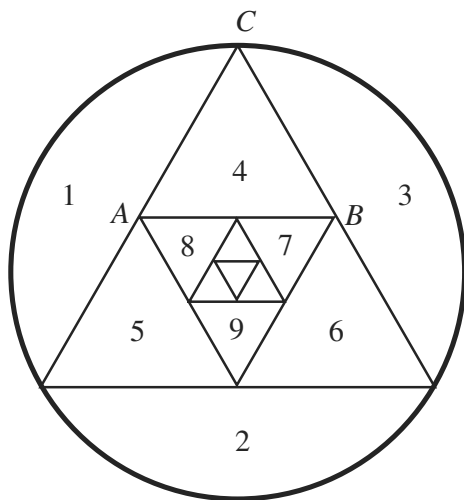


Fig. 91. The first regions to be cut off the disc  $D$ ; to save room, we use 1, 2, ... to denote them.

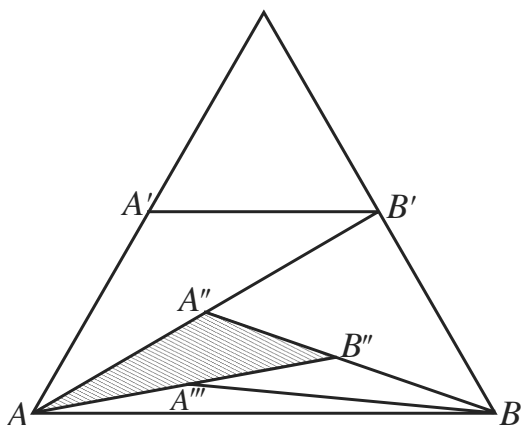


Fig. 92. When the shaded triangle  $AA'B''$  is cut off a convex domain  $K_\gamma$ , the part of this convex domain above the line  $AB''$  is only the triangle  $AA'B''$ . Every new point is the midpoint of the segment containing it:  $A'$  is the midpoint of  $AC$ ,  $B'$  is the midpoint of  $BC$ ,  $A''$  is the midpoint of  $AB'$ ,  $B''$  is the midpoint of  $A'B$ , etc.

furthermore  $T$  has diameter  $AB$  and is glued to the rest of  $K_\gamma$  along  $AB$ . Thus, the line  $AB$  cuts  $K_\gamma$  into two parts, the triangle  $T = ABC$  and the rest of  $K_\gamma$ . For example, if  $K_\gamma$  is the set obtained from  $D$  by cutting off the half-moons 1, 2 and 3, as in Figure 91, then  $ABC$  is such a triangle. A simple scheme for cutting is based on consecutively halving the sides (never  $AB$ , of course!), as in Figure 92. Thus, the first cut is the triangle  $A'B'C$ , then comes  $AA'B'$ , then  $A''B'B'$ , then  $AA'B''$ , etc. It is immediate that each cut has diameter at most  $\sqrt{3}/2$  times  $AB$ , the diameter of the triangle  $ABC$ .

Clearly, a similar (or even simpler) scheme works for a half-moon. It is immediate that the sets  $K_\gamma$  constructed in this way have properties (i)–(v) in the Claim. This proves our claim and so our result.  $\square$

**Notes.** This problem is clearly an up-market version of Problem 1 about the Christian and the lion. The lion lays down straight lines for the Christian to run along, and the Christian loses if he finds himself converging to a point.

In fact, the problem was posed and solved by Malý and Zelený for a ‘proper’ mathematical reason: to give an alternative and surprising proof of Buczolic’s result that there is a differentiable function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $\nabla f(0) = 0$  and  $|\nabla f| \geq 1$  a.e. With this result Buczolic solved Weil’s *gradient problem* that had been open for almost thirty years.

The gradient problem, posed by Weil in the 1960s, is the following. Suppose that  $n \geq 2$ ,  $G \subset \mathbb{R}^n$  is open and  $f : G \rightarrow \mathbb{R}$  is differentiable, so that  $\nabla f$  maps  $G$



into  $\mathbb{R}^n$ . Is it true that for every open set  $\Omega \subset \mathbb{R}^n$  the set  $(\nabla f)^{-1}(\Omega)$  is either empty or has positive Lebesgue measure? For  $n = 1$  this is true and is called the Denjoy–Clarkson property. Before Buczolich solved the gradient problem in the negative, he had published several results stating that under more stringent conditions the answer is in the affirmative.

The Malý and Zelený result reached me via Mark Walters from András Zsák.

Z. Buczolich, The  $n$ -dimensional gradient has the 1-dimensional Denjoy–Clarkson property, *Real Anal. Exchange* **18** (1992/93), 221–224.

Z. Buczolich, Another note on the gradient problem of C.E. Weil, *Real Anal. Exchange* **22** (1996/97), 775–784.

Z. Buczolich, Solution to the gradient problem of C.E. Weil, *Rev. Mat. Iberoamericana* **21** (2005), 889–910.

J.A. Clarkson, A property of derivatives, *Bull. Amer. Math. Soc.* **53** (1947), 124–125.

A. Denjoy, Sur une propriété des fonctions dérivée, *Enseignement Mathématique* **18** (1916), 320–328.

C.E. Weil, On properties of derivatives, *Trans. Amer. Math. Soc.* **114** (1965), 363–376.

